JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Convergent Sequences and the Limit of Sequences

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Summary. The article contains definitions and same basic properties of bounded sequences (above and below), convergent sequences and the limit of sequences. In the article there are some properties of real numbers useful in the other theorems of this article.

MML Identifier: SEQ_2.

WWW: http://mizar.org/JFM/Vol1/seq_2.html

The articles [1], [6], [3], [5], [7], [2], and [4] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: n, m are natural numbers, r, r_1, p, g_1, g are real numbers, s_1, s'_1, s_2 are sequences of real numbers, y is a set, and f is a real-yielding function.

We now state several propositions:

(3)[1] If $0 < g$, then $0 < \frac{g}{2}$ and $0 < \frac{g}{4}$.

(4) If $0 < g$, then $\frac{g}{2} < g$.

(6)[2] If $0 < g$ and $0 < p$, then $0 < \frac{g}{p}$.

(7) If $0 \leq g$ and $0 \leq r$ and $g < g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$.

(9)[3] $-g < r$ and $r < g$ iff $|r| < g$.

(10) If $0 < r_1$ and $r_1 < r$ and $0 < g$, then $\frac{g}{r} < \frac{g}{r_1}$.

(11) If $g \neq 0$ and $r \neq 0$, then $|g^{-1} - r^{-1}| = \frac{|g-r|}{|g| \cdot |r|}$.

Let f be a real-yielding function. We say that f is upper bounded if and only if:

(Def. 1) There exists r such that for every y such that $y \in \mathrm{dom}\, f$ holds $f(y) < r$.

We say that f is lower bounded if and only if:

(Def. 2) There exists r such that for every y such that $y \in \mathrm{dom}\, f$ holds $r < f(y)$.

Let us consider s_1. Let us observe that s_1 is upper bounded if and only if:

(Def. 3) There exists r such that for every n holds $s_1(n) < r$.

Let us observe that s_1 is lower bounded if and only if:

[1] The propositions (1) and (2) have been removed.
[2] The proposition (5) has been removed.
[3] The proposition (8) has been removed.

(Def. 4) There exists r such that for every n holds $r < s_1(n)$.

Let us consider f. We say that f is bounded if and only if:

(Def. 5) f is upper bounded and lower bounded.

Let us mention that every real-yielding function which is bounded is also upper bounded and lower bounded and every real-yielding function which is upper bounded and lower bounded is also bounded.

The following two propositions are true:

(15)[4] s_1 is bounded iff there exists r such that $0 < r$ and for every n holds $|s_1(n)| < r$.

(16) For every n there exists r such that $0 < r$ and for every m such that $m \leq n$ holds $|s_1(m)| < r$.

Let us consider s_1. We say that s_1 is convergent if and only if:

(Def. 6) There exists g such that for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - g| < p$.

Let us consider s_1. Let us assume that s_1 is convergent. The functor $\lim s_1$ yields a real number and is defined as follows:

(Def. 7) For every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$.

Let us consider s_1. Then $\lim s_1$ is a real number.

Next we state a number of propositions:

(19)[5] If s_1 is convergent and s'_1 is convergent, then $s_1 + s'_1$ is convergent.

(20) If s_1 is convergent and s'_1 is convergent, then $\lim(s_1 + s'_1) = \lim s_1 + \lim s'_1$.

(21) If s_1 is convergent, then $r\, s_1$ is convergent.

(22) If s_1 is convergent, then $\lim(r\, s_1) = r \cdot \lim s_1$.

(23) If s_1 is convergent, then $-s_1$ is convergent.

(24) If s_1 is convergent, then $\lim(-s_1) = -\lim s_1$.

(25) If s_1 is convergent and s'_1 is convergent, then $s_1 - s'_1$ is convergent.

(26) If s_1 is convergent and s'_1 is convergent, then $\lim(s_1 - s'_1) = \lim s_1 - \lim s'_1$.

(27) If s_1 is convergent, then s_1 is bounded.

(28) If s_1 is convergent and s'_1 is convergent, then $s_1\, s'_1$ is convergent.

(29) If s_1 is convergent and s'_1 is convergent, then $\lim(s_1\, s'_1) = \lim s_1 \cdot \lim s'_1$.

(30) If s_1 is convergent, then if $\lim s_1 \neq 0$, then there exists n such that for every m such that $n \leq m$ holds $\frac{|\lim s_1|}{2} < |s_1(m)|$.

(31) If s_1 is convergent and for every n holds $0 \leq s_1(n)$, then $0 \leq \lim s_1$.

(32) If s_1 is convergent and s'_1 is convergent and for every n holds $s_1(n) \leq s'_1(n)$, then $\lim s_1 \leq \lim s'_1$.

(33) If s_1 is convergent and s'_1 is convergent and for every n holds $s_1(n) \leq s_2(n)$ and $s_2(n) \leq s'_1(n)$ and $\lim s_1 = \lim s'_1$, then s_2 is convergent.

[4] The propositions (12)–(14) have been removed.

[5] The propositions (17) and (18) have been removed.

(34) If s_1 is convergent and s'_1 is convergent and for every n holds $s_1(n) \leq s_2(n)$ and $s_2(n) \leq s'_1(n)$ and $\lim s_1 = \lim s'_1$, then $\lim s_2 = \lim s_1$.

(35) If s_1 is convergent and $\lim s_1 \neq 0$ and s_1 is non-zero, then $s_1{}^{-1}$ is convergent.

(36) If s_1 is convergent and $\lim s_1 \neq 0$ and s_1 is non-zero, then $\lim(s_1{}^{-1}) = (\lim s_1)^{-1}$.

(37) If s'_1 is convergent and s_1 is convergent and $\lim s_1 \neq 0$ and s_1 is non-zero, then s'_1/s_1 is convergent.

(38) If s'_1 is convergent and s_1 is convergent and $\lim s_1 \neq 0$ and s_1 is non-zero, then $\lim(s'_1/s_1) = \frac{\lim s'_1}{\lim s_1}$.

(39) If s_1 is convergent and s_2 is bounded and $\lim s_1 = 0$, then $s_1\, s_2$ is convergent.

(40) If s_1 is convergent and s_2 is bounded and $\lim s_1 = 0$, then $\lim(s_1\, s_2) = 0$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[4] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[5] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[6] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[7] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received July 4, 1989

Published January 2, 2004