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The Sum and Product of Finite Sequences of Real Numbers

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Summary. Some operations on the set of n-tuples of real numbers are introduced. Addition, difference of such n-tuples, complement of a n-tuple and multiplication of these by real numbers are defined. In these definitions more general properties of binary operations applied to finite sequences from [9] are used. Then the fact that certain properties are satisfied by those operations is demonstrated directly from [9]. Moreover some properties can be recognized as being those of real vector space. Multiplication of n-tuples of real numbers and square power of n-tuple of real numbers using for notation of some properties of finite sums and products of real numbers are defined, followed by definitions of the finite sum and product of n-tuples of real numbers using notions and properties introduced in [11]. A number of propositions and theorems on sum and product of finite sequences of real numbers are proved. As additional properties there are proved some properties of real numbers and set representations of binary operations on real numbers.

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The articles [17], [21], [8], [2], [18], [12], [1], [19], [22], [5], [7], [6], [4], [15], [14], [16], [3], [10], [20], [13], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: i, j, k denote natural numbers, x, r, r_1, r_2, r_3 denote elements of $\mathbb{R}$, F, F_1, F_2 denote finite sequences of elements of $\mathbb{R}$, and R, R_1, R_2, R_3 denote elements of $\mathbb{R}^i$.

One can prove the following propositions:

(3)[1] 0 is a unity w.r.t. $+_{\mathbb{R}}$.

(4) $\mathbf{1}_{+_{\mathbb{R}}} = 0$.

(5) $+_{\mathbb{R}}$ has a unity.

(6) $+_{\mathbb{R}}$ is commutative.

(7) $+_{\mathbb{R}}$ is associative.

The binary operation $-_{\mathbb{R}}$ on $\mathbb{R}$ is defined by:

(Def. 1) $-_{\mathbb{R}} = +_{\mathbb{R}} \circ (\mathrm{id}_{\mathbb{R}}, -_{\mathbb{R}})$.

The following proposition is true

[1] The propositions (1) and (2) have been removed.

(9)[2] $-_{\mathbb{R}}(r_1, r_2) = r_1 - r_2$.

The unary operation $\mathrm{sqr}_{\mathbb{R}}$ on $\mathbb{R}$ is defined by:

(Def. 2) For every r holds $\mathrm{sqr}_{\mathbb{R}}(r) = r^{\mathbf{2}}$.

We now state several propositions:

(11)[3] $\cdot_{\mathbb{R}}$ is commutative.

(12) $\cdot_{\mathbb{R}}$ is associative.

(13) 1 is a unity w.r.t. $\cdot_{\mathbb{R}}$.

(14) $\mathbf{1}_{\cdot_{\mathbb{R}}} = 1$.

(15) $\cdot_{\mathbb{R}}$ has a unity.

(16) $\cdot_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

(17) $\mathrm{sqr}_{\mathbb{R}}$ is distributive w.r.t. $\cdot_{\mathbb{R}}$.

Let x be a real number. The functor $\cdot^{x}_{\mathbb{R}}$ yields a unary operation on $\mathbb{R}$ and is defined as follows:

(Def. 3) $\cdot^{x}_{\mathbb{R}} = (\cdot_{\mathbb{R}})^{\circ}(x, \mathrm{id}_{\mathbb{R}})$.

We now state several propositions:

(19)[4] $\cdot^{r}_{\mathbb{R}}(x) = r \cdot x$.

(20) $\cdot^{r}_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

(21) $-_{\mathbb{R}}$ is an inverse operation w.r.t. $+_{\mathbb{R}}$.

(22) $+_{\mathbb{R}}$ has an inverse operation.

(23) The inverse operation w.r.t. $+_{\mathbb{R}} = -_{\mathbb{R}}$.

(24) $-_{\mathbb{R}}$ is distributive w.r.t. $+_{\mathbb{R}}$.

Let us consider F_1, F_2. The functor $F_1 + F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 4) $F_1 + F_2 = (+_{\mathbb{R}})^{\circ}(F_1, F_2)$.

Let us observe that the functor $F_1 + F_2$ is commutative.

We now state the proposition

(26)[5] If $i \in \mathrm{dom}(F_1 + F_2)$, then $(F_1 + F_2)(i) = F_1(i) + F_2(i)$.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{R}^{i}$.

Next we state several propositions:

(27) $(R_1 + R_2)(j) = R_1(j) + R_2(j)$.

(28) $\varepsilon_{\mathbb{R}} + F = \varepsilon_{\mathbb{R}}$.

(29) $\langle r_1 \rangle + \langle r_2 \rangle = \langle r_1 + r_2 \rangle$.

(30) $i \mapsto r_1 + i \mapsto r_2 = i \mapsto (r_1 + r_2)$.

[2] The proposition (8) has been removed.
[3] The proposition (10) has been removed.
[4] The proposition (18) has been removed.
[5] The proposition (25) has been removed.

(32)[6] $R_1 + (R_2 + R_3) = (R_1 + R_2) + R_3$.

(33) $R + i \mapsto (0$ **qua** real number$) = R$.

Let us consider F. The functor $-F$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 5) $-F = -_{\mathbb{R}} \cdot F$.

Let us note that the functor $-F$ is involutive.

We now state two propositions:

(34) $\operatorname{dom} F = \operatorname{dom}(-F)$.

(35) $(-F)(i) = -F(i)$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{R}^i$.

One can prove the following propositions:

(36) If $r = R(j)$, then $(-R)(j) = -r$.

(37) $-\varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(38) $-\langle r \rangle = \langle -r \rangle$.

(39) $-i \mapsto r = i \mapsto (-r)$.

(40) $R + -R = i \mapsto 0$.

(41) If $R_1 + R_2 = i \mapsto 0$, then $R_1 = -R_2$.

(43)[7] If $-R_1 = -R_2$, then $R_1 = R_2$.

(44) If $R_1 + R = R_2 + R$, then $R_1 = R_2$.

(45) $-(R_1 + R_2) = -R_1 + -R_2$.

Let us consider F_1, F_2. The functor $F_1 - F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 6) $F_1 - F_2 = (-_{\mathbb{R}})^{\circ}(F_1, F_2)$.

We now state the proposition

(47)[8] If $i \in \operatorname{dom}(F_1 - F_2)$, then $(F_1 - F_2)(i) = F_1(i) - F_2(i)$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{R}^i$.

Next we state a number of propositions:

(48) $(R_1 - R_2)(j) = R_1(j) - R_2(j)$.

(49) $\varepsilon_{\mathbb{R}} - F = \varepsilon_{\mathbb{R}}$ and $F - \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(50) $\langle r_1 \rangle - \langle r_2 \rangle = \langle r_1 - r_2 \rangle$.

(51) $i \mapsto r_1 - i \mapsto r_2 = i \mapsto (r_1 - r_2)$.

(52) $R_1 - R_2 = R_1 + -R_2$.

(53) $R - i \mapsto (0$ **qua** real number$) = R$.

(54) $i \mapsto (0$ **qua** real number$) - R = -R$.

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[6] The proposition (31) has been removed.
[7] The proposition (42) has been removed.
[8] The proposition (46) has been removed.

(55) $R_1 - -R_2 = R_1 + R_2$.

(56) $-(R_1 - R_2) = R_2 - R_1$.

(57) $-(R_1 - R_2) = -R_1 + R_2$.

(58) $R - R = i \mapsto 0$.

(59) If $R_1 - R_2 = i \mapsto 0$, then $R_1 = R_2$.

(60) $R_1 - R_2 - R_3 = R_1 - (R_2 + R_3)$.

(61) $R_1 + (R_2 - R_3) = (R_1 + R_2) - R_3$.

(62) $R_1 - (R_2 - R_3) = (R_1 - R_2) + R_3$.

(63) $R_1 = (R_1 + R) - R$.

(64) $R_1 = (R_1 - R) + R$.

Let r be a real number and let us consider F. The functor $r \cdot F$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 7) $r \cdot F = \cdot^{r}_{\mathbb{R}} \cdot F$.

One can prove the following two propositions:

(65) $\mathrm{dom}(r \cdot F) = \mathrm{dom} F$.

(66) $(r \cdot F)(i) = r \cdot F(i)$.

Let us consider i, let r be a real number, and let us consider R. Then $r \cdot R$ is an element of $\mathbb{R}^i$.
One can prove the following propositions:

(67) $(r \cdot R)(j) = r \cdot R(j)$.

(68) $r \cdot \varepsilon_{\mathbb{R}} = \varepsilon_{\mathbb{R}}$.

(69) $r \cdot \langle r_1 \rangle = \langle r \cdot r_1 \rangle$.

(70) $r_1 \cdot (i \mapsto r_2) = i \mapsto (r_1 \cdot r_2)$.

(71) $(r_1 \cdot r_2) \cdot R = r_1 \cdot (r_2 \cdot R)$.

(72) $(r_1 + r_2) \cdot R = r_1 \cdot R + r_2 \cdot R$.

(73) $r \cdot (R_1 + R_2) = r \cdot R_1 + r \cdot R_2$.

(74) $1 \cdot R = R$.

(75) $0 \cdot R = i \mapsto 0$.

(76) $(-1) \cdot R = -R$.

Let us consider F. The functor ${}^2F$ yielding a finite sequence of elements of $\mathbb{R}$ is defined by:

(Def. 8) ${}^2F = \mathrm{sqr}_{\mathbb{R}} \cdot F$.

Next we state two propositions:

(77) $\mathrm{dom}\,{}^2F = \mathrm{dom} F$.

(78) $({}^2F)(i) = F(i)^2$.

Let us consider i, R. Then ${}^2R$ is an element of $\mathbb{R}^i$.
The following propositions are true:

(79) $({}^{2}R)(j) = R(j)^{\mathbf{2}}$.

(80) ${}^{2}(\varepsilon_{\mathbb{R}}) = \varepsilon_{\mathbb{R}}$.

(81) ${}^{2}\langle r \rangle = \langle r^{\mathbf{2}} \rangle$.

(82) ${}^{2}(i \mapsto r) = i \mapsto r^{\mathbf{2}}$.

(83) ${}^{2}{-R} = {}^{2}R$.

(84) ${}^{2}(r \cdot R) = r^{\mathbf{2}} \cdot {}^{2}R$.

Let us consider F_1, F_2. The functor $F_1 \bullet F_2$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 9) $F_1 \bullet F_2 = (\cdot_{\mathbb{R}})^{\circ}(F_1, F_2)$.

Let us note that the functor $F_1 \bullet F_2$ is commutative.

The following proposition is true

(86)[9] If $i \in \mathrm{dom}(F_1 \bullet F_2)$, then $(F_1 \bullet F_2)(i) = F_1(i) \cdot F_2(i)$.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{R}^{i}$.

Next we state a number of propositions:

(87) $(R_1 \bullet R_2)(j) = R_1(j) \cdot R_2(j)$.

(88) $\varepsilon_{\mathbb{R}} \bullet F = \varepsilon_{\mathbb{R}}$.

(89) $\langle r_1 \rangle \bullet \langle r_2 \rangle = \langle r_1 \cdot r_2 \rangle$.

(91)[10] $R_1 \bullet (R_2 \bullet R_3) = (R_1 \bullet R_2) \bullet R_3$.

(92) $i \mapsto r \bullet R = r \cdot R$.

(93) $i \mapsto r_1 \bullet i \mapsto r_2 = i \mapsto (r_1 \cdot r_2)$.

(94) $r \cdot (R_1 \bullet R_2) = r \cdot R_1 \bullet R_2$.

(96)[11] $r \cdot R = i \mapsto r \bullet R$.

(97) ${}^{2}R = R \bullet R$.

(98) ${}^{2}(R_1 + R_2) = {}^{2}R_1 + 2 \cdot (R_1 \bullet R_2) + {}^{2}R_2$.

(99) ${}^{2}(R_1 - R_2) = ({}^{2}R_1 - 2 \cdot (R_1 \bullet R_2)) + {}^{2}R_2$.

(100) ${}^{2}(R_1 \bullet R_2) = {}^{2}R_1 \bullet {}^{2}R_2$.

Let F be a finite sequence of elements of $\mathbb{R}$. The functor $\sum F$ yielding a real number is defined by:

(Def. 10) $\sum F = +_{\mathbb{R}} \circledast F$.

The following propositions are true:

(102)[12] $\sum(\varepsilon_{\mathbb{R}}) = 0$.

(103) $\sum \langle r \rangle = r$.

(104) $\sum(F \frown \langle r \rangle) = \sum F + r$.

[9] The proposition (85) has been removed.
[10] The proposition (90) has been removed.
[11] The proposition (95) has been removed.
[12] The proposition (101) has been removed.

(105) $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

(106) $\sum(\langle r \rangle \frown F) = r + \sum F$.

(107) $\sum\langle r_1, r_2 \rangle = r_1 + r_2$.

(108) $\sum\langle r_1, r_2, r_3 \rangle = r_1 + r_2 + r_3$.

(109) For every element R of $\mathbb{R}^0$ holds $\sum R = 0$.

(110) $\sum(i \mapsto r) = i \cdot r$.

(111) $\sum(i \mapsto$ (0 **qua** real number)) $= 0$.

(112) If for every j such that $j \in \mathrm{Seg}\, i$ holds $R_1(j) \leq R_2(j)$, then $\sum R_1 \leq \sum R_2$.

(113) If for every j such that $j \in \mathrm{Seg}\, i$ holds $R_1(j) \leq R_2(j)$ and there exists j such that $j \in \mathrm{Seg}\, i$ and $R_1(j) < R_2(j)$, then $\sum R_1 < \sum R_2$.

(114) If for every i such that $i \in \mathrm{dom} F$ holds $0 \leq F(i)$, then $0 \leq \sum F$.

(115) If for every i such that $i \in \mathrm{dom} F$ holds $0 \leq F(i)$ and there exists i such that $i \in \mathrm{dom} F$ and $0 < F(i)$, then $0 < \sum F$.

(116) $0 \leq \sum^2 F$.

(117) $\sum(r \cdot F) = r \cdot \sum F$.

(118) $\sum(-F) = -\sum F$.

(119) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(120) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

(121) If $\sum^2 R = 0$, then $R = i \mapsto 0$.

(122) $(\sum(R_1 \bullet R_2))^{\mathbf{2}} \leq \sum^2 R_1 \cdot \sum^2 R_2$.

Let F be a finite sequence of elements of $\mathbb{R}$. The functor $\prod F$ yields a real number and is defined as follows:

(Def. 11) $\prod F = \cdot_{\mathbb{R}} \circledast F$.

Next we state a number of propositions:

(124)[13] $\prod(\varepsilon_{\mathbb{R}}) = 1$.

(125) $\prod\langle r \rangle = r$.

(126) $\prod(F \frown \langle r \rangle) = \prod F \cdot r$.

(127) $\prod(F_1 \frown F_2) = \prod F_1 \cdot \prod F_2$.

(128) $\prod(\langle r \rangle \frown F) = r \cdot \prod F$.

(129) $\prod\langle r_1, r_2 \rangle = r_1 \cdot r_2$.

(130) $\prod\langle r_1, r_2, r_3 \rangle = r_1 \cdot r_2 \cdot r_3$.

(131) For every element R of $\mathbb{R}^0$ holds $\prod R = 1$.

(132) $\prod(i \mapsto$ (1 **qua** real number)) $= 1$.

(133) There exists k such that $k \in \mathrm{dom} F$ and $F(k) = 0$ iff $\prod F = 0$.

[13] The proposition (123) has been removed.

(134) $\prod((i+j)\mapsto r)=\prod(i\mapsto r)\cdot\prod(j\mapsto r)$.

(135) $\prod((i\cdot j)\mapsto r)=\prod(j\mapsto\prod(i\mapsto r))$.

(136) $\prod(i\mapsto(r_1\cdot r_2))=\prod(i\mapsto r_1)\cdot\prod(i\mapsto r_2)$.

(137) $\prod(R_1\bullet R_2)=\prod R_1\cdot\prod R_2$.

(138) $\prod(r\cdot R)=\prod(i\mapsto r)\cdot\prod R$.

(139) $\prod^2 R=(\prod R)^{\mathbf{2}}$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[7] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[8] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[9] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseqop.html.

[10] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[11] Czesław Byliński. Semigroup operations on finite subsets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/setwop_2.html.

[12] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[14] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[15] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[16] Andrzej Trybulec. Semilattice operations on finite subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setwiseo.html.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[18] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/square_1.html.

[20] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[21] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[22] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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