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Subspaces and Cosets of Subspaces in Real Linear Space

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Summary. The following notions are introduced in the article: subspace of a real linear space, zero subspace and improper subspace, coset of a subspace. The relation of a subset of the vectors being linearly closed is also introduced. Basic theorems concerning those notions are proved in the article.

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The articles [4], [3], [8], [6], [5], [1], [9], [2], and [7] provide the notation and terminology for this paper.

For simplicity, we follow the rules: V, X, Y denote real linear spaces, u, v, v_1, v_2 denote vectors of V, a denotes a real number, V_1, V_2, V_3 denote subsets of V, and x denotes a set.

Let us consider V and let us consider V_1. We say that V_1 is linearly closed if and only if:

(Def. 1) For all v, u such that $v \in V_1$ and $u \in V_1$ holds $v+u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

We now state several propositions:

(4)[1] If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v-u \in V_1$.

(7) $\{0_V\}$ is linearly closed.

(8) If the carrier of $V = V_1$, then V_1 is linearly closed.

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v+u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider V. A real linear space is said to be a subspace of V if it satisfies the conditions (Def. 2).

(Def. 2)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it $=$ the zero of V,

(iii) the addition of it $=$ (the addition of V) $\restriction [\!:$the carrier of it, the carrier of it$:\!]$, and

(iv) the external multiplication of it $=$ (the external multiplication of V) $\restriction [\!:\mathbb{R}$, the carrier of it$:\!]$.

[1] The propositions (1)–(3) have been removed.

We adopt the following rules: W, W_1, W_2 denote subspaces of V and w, w_1, w_2 denote vectors of W.

One can prove the following propositions:

(16)[2] If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $0_W = 0_V$.

(20) $0_{(W_1)} = 0_{(W_2)}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $0_V \in W$.

(26) $0_{(W_1)} \in W_2$.

(27) $0_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

In the sequel D is a non empty set, d_1 is an element of D, A is a binary operation on D, and M is a function from $[:\mathbb{R}, D:]$ into D.

The following propositions are true:

(32) Suppose $V_1 = D$ and $d_1 = 0_V$ and $A =$ (the addition of $V) \restriction [:V_1, V_1:]$ and $M =$ (the external multiplication of $V) \restriction [:\mathbb{R}, V_1:]$. Then $\langle D, d_1, A, M \rangle$ is a subspace of V.

(33) V is a subspace of V.

(34) For all strict real linear spaces V, X such that V is a subspace of X and X is a subspace of V holds $V = X$.

(35) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(36) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a subspace of W_2.

(37) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

Let us consider V. Note that there exists a subspace of V which is strict.

One can prove the following propositions:

(38) For all strict subspaces W_1, W_2 of V such that the carrier of $W_1 =$ the carrier of W_2 holds $W_1 = W_2$.

(39) For all strict subspaces W_1, W_2 of V such that for every v holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

[2] The propositions (11)–(15) have been removed.

(40) Let V be a strict real linear space and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(41) Let V be a strict real linear space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(42) If the carrier of $W = V_1$, then V_1 is linearly closed.

(43) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

Let us consider V. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined as follows:

(Def. 3) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider V. The functor Ω_V yielding a strict subspace of V is defined by:

(Def. 4) $\Omega_V =$ the RLS structure of V.

We now state several propositions:

(48)[3] $\mathbf{0}_W = \mathbf{0}_V$.

(49) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(50) $\mathbf{0}_W$ is a subspace of V.

(51) $\mathbf{0}_V$ is a subspace of W.

(52) $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(54)[4] Every strict real linear space V is a subspace of Ω_V.

Let us consider V and let us consider v, W. The functor $v+W$ yields a subset of V and is defined as follows:

(Def. 5) $v+W = \{v+u : u \in W\}$.

Let us consider V and let us consider W. A subset of V is called a coset of W if:

(Def. 6) There exists v such that it $= v+W$.

In the sequel B, C denote cosets of W.
Next we state a number of propositions:

(58)[5] $0_V \in v+W$ iff $v \in W$.

(59) $v \in v+W$.

(60) $0_V + W =$ the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of V.

(63) $0_V \in v+W$ iff $v+W =$ the carrier of W.

(64) $v \in W$ iff $v+W =$ the carrier of W.

(65) If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(66) If $a \neq 0$ and $a \cdot v + W =$ the carrier of W, then $v \in W$.

[3] The propositions (44)–(47) have been removed.
[4] The proposition (53) has been removed.
[5] The propositions (55)–(57) have been removed.

(67) $v \in W$ iff $-v+W =$ the carrier of W.

(68) $u \in W$ iff $v+W = v+u+W$.

(69) $u \in W$ iff $v+W = (v-u)+W$.

(70) $v \in u+W$ iff $u+W = v+W$.

(71) $v+W = -v+W$ iff $v \in W$.

(72) If $u \in v_1+W$ and $u \in v_2+W$, then $v_1+W = v_2+W$.

(73) If $u \in v+W$ and $u \in -v+W$, then $v \in W$.

(74) If $a \neq 1$ and $a \cdot v \in v+W$, then $v \in W$.

(75) If $v \in W$, then $a \cdot v \in v+W$.

(76) $-v \in v+W$ iff $v \in W$.

(77) $u+v \in v+W$ iff $u \in W$.

(78) $v-u \in v+W$ iff $u \in W$.

(79) $u \in v+W$ iff there exists v_1 such that $v_1 \in W$ and $u = v+v_1$.

(80) $u \in v+W$ iff there exists v_1 such that $v_1 \in W$ and $u = v-v_1$.

(81) There exists v such that $v_1 \in v+W$ and $v_2 \in v+W$ iff $v_1 - v_2 \in W$.

(82) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v+v_1 = u$.

(83) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v-v_1 = u$.

(84) For all strict subspaces W_1, W_2 of V holds $v+W_1 = v+W_2$ iff $W_1 = W_2$.

(85) For all strict subspaces W_1, W_2 of V such that $v+W_1 = u+W_2$ holds $W_1 = W_2$.

(86) C is linearly closed iff $C =$ the carrier of W.

(87) For all strict subspaces W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1 = C_2$ holds $W_1 = W_2$.

(88) $\{v\}$ is a coset of $\mathbf{0}_V$.

(89) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(90) The carrier of W is a coset of W.

(91) The carrier of V is a coset of $\boldsymbol{\Omega}_V$.

(92) If V_1 is a coset of $\boldsymbol{\Omega}_V$, then $V_1 =$ the carrier of V.

(93) $0_V \in C$ iff $C =$ the carrier of W.

(94) $u \in C$ iff $C = u+W$.

(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.

(96) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.

(97) There exists C such that $v_1 \in C$ and $v_2 \in C$ iff $v_1 - v_2 \in W$.

(98) If $u \in B$ and $u \in C$, then $B = C$.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[5] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[6] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[7] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[9] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

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