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The Lattice of Real Numbers. The Lattice of Real Functions

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Summary. A proof of the fact, that $\langle\mathbb{R},\max,\min\rangle$ is a lattice (real lattice). Some basic properties (real lattice is distributive and modular) of it are proved. The same is done for the set $\mathbb{R}^A$ with operations: $\max(f(A))$ and $\min(f(A))$, where $\mathbb{R}^A$ means the set of all functions from A (being non-empty set) to $\mathbb{R}$, f is just such a function.

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The articles [7], [5], [6], [9], [1], [3], [8], [2], and [4] provide the notation and terminology for this paper.

In this paper x, y denote real numbers.

The binary operation $\min_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:

(Def. 1) $\min_{\mathbb{R}}(x, y) = \min(x,y)$.

The binary operation $\max_{\mathbb{R}}$ on $\mathbb{R}$ is defined as follows:

(Def. 2) $\max_{\mathbb{R}}(x, y) = \max(x,y)$.

The strict lattice structure $\mathbb{R}_{\rm L}$ is defined as follows:

(Def. 4)[1] $\mathbb{R}_{\rm L} = \langle\mathbb{R},\max_{\mathbb{R}},\min_{\mathbb{R}}\rangle$.

Let us mention that every element of $\mathbb{R}_{\rm L}$ is real.

One can verify that $\mathbb{R}_{\rm L}$ is non empty.

Let us mention that $\mathbb{R}_{\rm L}$ is lattice-like.

In the sequel p, q, r are elements of $\mathbb{R}_{\rm L}$.

We now state several propositions:

(8)[2] $\max_{\mathbb{R}}(p, q) = \max_{\mathbb{R}}(q, p)$.

(9) $\min_{\mathbb{R}}(p, q) = \min_{\mathbb{R}}(q, p)$.

(10)(i) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, r), p)$,

(ii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), r)$,

(iii) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), r)$,

(iv) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, p), q)$,

(v) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(r, q), p)$, and

(vi) $\max_{\mathbb{R}}(p, \max_{\mathbb{R}}(q, r)) = \max_{\mathbb{R}}(\max_{\mathbb{R}}(p, r), q)$.

[1] The definition (Def. 3) has been removed.

[2] The propositions (1)–(7) have been removed.

(11)(i) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, r), p)$,

(ii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), r)$,

(iii) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), r)$,

(iv) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, p), q)$,

(v) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(r, q), p)$, and

(vi) $\min_{\mathbb{R}}(p, \min_{\mathbb{R}}(q, r)) = \min_{\mathbb{R}}(\min_{\mathbb{R}}(p, r), q)$.

(12) $\max_{\mathbb{R}}(\min_{\mathbb{R}}(p, q), q) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(p, q)) = q$ and $\max_{\mathbb{R}}(q, \min_{\mathbb{R}}(q, p)) = q$ and $\max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), q) = q$.

(13) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(q, p)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(p, q), q) = q$ and $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, q)) = q$ and $\min_{\mathbb{R}}(\max_{\mathbb{R}}(q, p), q) = q$.

(14) $\min_{\mathbb{R}}(q, \max_{\mathbb{R}}(p, r)) = \max_{\mathbb{R}}(\min_{\mathbb{R}}(q, p), \min_{\mathbb{R}}(q, r))$.

One can verify that $\mathbb{R}_{\rm L}$ is distributive.

In the sequel A is a non empty set and f, g, h are elements of $\mathbb{R}^A$.

Let us consider A. The functor $\mathbf{max}_{\mathbb{R}^A}$ yielding a binary operation on $\mathbb{R}^A$ is defined by:

(Def. 5) $\mathbf{max}_{\mathbb{R}^A}(f, g) = (\max_{\mathbb{R}})^{\circ}(f, g)$.

The functor $\mathbf{min}_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:

(Def. 6) $\mathbf{min}_{\mathbb{R}^A}(f, g) = (\min_{\mathbb{R}})^{\circ}(f, g)$.

The following propositions are true:

(20)[3] $\mathbf{max}_{\mathbb{R}^A}(f, g) = \mathbf{max}_{\mathbb{R}^A}(g, f)$.

(21) $\mathbf{min}_{\mathbb{R}^A}(f, g) = \mathbf{min}_{\mathbb{R}^A}(g, f)$.

(22) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(f, g), h) = \mathbf{max}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(g, h))$.

(23) $\mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(f, g), h) = \mathbf{min}_{\mathbb{R}^A}(f, \mathbf{min}_{\mathbb{R}^A}(g, h))$.

(24) $\mathbf{max}_{\mathbb{R}^A}(f, \mathbf{min}_{\mathbb{R}^A}(f, g)) = f$.

(25) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(f, g), f) = f$.

(26) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(g, f), f) = f$.

(27) $\mathbf{max}_{\mathbb{R}^A}(f, \mathbf{min}_{\mathbb{R}^A}(g, f)) = f$.

(28) $\mathbf{min}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(f, g)) = f$.

(29) $\mathbf{min}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(g, f)) = f$.

(30) $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(g, f), f) = f$.

(31) $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(f, g), f) = f$.

(32) $\mathbf{min}_{\mathbb{R}^A}(f, \mathbf{max}_{\mathbb{R}^A}(g, h)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(f, g), \mathbf{min}_{\mathbb{R}^A}(f, h))$.

Let us consider A and let x be an element of $\langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$. The functor $^{@}x$ yielding an element of $\mathbb{R}^A$ is defined as follows:

(Def. 9)[4] $^{@}x = x$.

Let us consider A. The functor $\mathbb{R}^A_{\rm L}$ yields a strict lattice and is defined by:

[3] The propositions (15)–(19) have been removed.

[4] The definitions (Def. 7) and (Def. 8) have been removed.

(Def. 10) $\mathbb{R}^A_{\rm L} = \langle \mathbb{R}^A, \mathbf{max}_{\mathbb{R}^A}, \mathbf{min}_{\mathbb{R}^A} \rangle$.

In the sequel p, q, r are elements of $\mathbb{R}^A_{\rm L}$.
One can prove the following propositions:

(40)[5] $\mathbf{max}_{\mathbb{R}^A}(p, q) = \mathbf{max}_{\mathbb{R}^A}(q, p)$.

(41) $\mathbf{min}_{\mathbb{R}^A}(p, q) = \mathbf{min}_{\mathbb{R}^A}(q, p)$.

(42)(i) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, r), p)$,
(ii) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, q), r)$,
(iii) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, p), r)$,
(iv) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(r, p), q)$,
(v) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(r, q), p)$, and
(vi) $\mathbf{max}_{\mathbb{R}^A}(p, \mathbf{max}_{\mathbb{R}^A}(q, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, r), q)$.

(43)(i) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, r), p)$,
(ii) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, q), r)$,
(iii) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), r)$,
(iv) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(r, p), q)$,
(v) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(r, q), p)$, and
(vi) $\mathbf{min}_{\mathbb{R}^A}(p, \mathbf{min}_{\mathbb{R}^A}(q, r)) = \mathbf{min}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, r), q)$.

(44) $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(p, q), q) = q$ and $\mathbf{max}_{\mathbb{R}^A}(q, \mathbf{min}_{\mathbb{R}^A}(p, q)) = q$ and $\mathbf{max}_{\mathbb{R}^A}(q, \mathbf{min}_{\mathbb{R}^A}(q, p)) = q$ and $\mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), q) = q$.

(45) $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(q, p)) = q$ and $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(p, q), q) = q$ and $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(p, q)) = q$ and $\mathbf{min}_{\mathbb{R}^A}(\mathbf{max}_{\mathbb{R}^A}(q, p), q) = q$.

(46) $\mathbf{min}_{\mathbb{R}^A}(q, \mathbf{max}_{\mathbb{R}^A}(p, r)) = \mathbf{max}_{\mathbb{R}^A}(\mathbf{min}_{\mathbb{R}^A}(q, p), \mathbf{min}_{\mathbb{R}^A}(q, r))$.

(47) $\mathbb{R}^A_{\rm L}$ is a distributive lattice.

REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funcsdom.html`.

[4] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/fraenkel.html`.

[5] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[6] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/square_1.html`.

[7] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[8] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[5] The propositions (33)–(39) have been removed.

[9] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

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