JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Equalities and Inequalities in Real Numbers

Andrzej Kondracki
Warsaw University

Summary. The aim of the article is to give a number of useful theorems concerning equalities and inequalities in real numbers. Some of the theorems are extensions of [2] theorems, others were found to be needed in practice.

MML Identifier: REAL_2.

WWW: http://mizar.org/JFM/Vol2/real_2.html

The articles [4], [1], [3], and [2] provide the notation and terminology for this paper.

In this paper a, b, d, e are real numbers.

The following propositions are true:

(63)[1] For all a, b there exists e such that $a = b - e$.

(64) For all a, b such that $a \neq 0$ and $b \neq 0$ there exists e such that $a = \frac{b}{e}$.

(105)[2] If $a + -b \leq 0$ or $b - a \geq 0$ or $b + -a \geq 0$ or $a - e \leq b + -e$ or $a + -e \leq b - e$ or $e - a \geq e - b$, then $a \leq b$.

(106) If $a + -b < 0$ or $b - a > 0$ or $-a + b > 0$ or $a - e < b + -e$ or $a + -e < b - e$ or $e - a > e - b$, then $a < b$.

(109)[3] If $a \leq -b$, then $a + b \leq 0$ and $-a \geq b$.

(110) If $a < -b$, then $a + b < 0$ and $-a > b$.

(111) If $-a \leq b$, then $a + b \geq 0$.

(112) If $-b < a$, then $a + b > 0$.

(117)[4] Suppose $b > 0$. Then

(i) if $\frac{a}{b} > 1$, then $a > b$,

(ii) if $\frac{a}{b} < 1$, then $a < b$,

(iii) if $\frac{a}{b} > -1$, then $a > -b$ and $b > -a$, and

(iv) if $\frac{a}{b} < -1$, then $a < -b$ and $b < -a$.

[1] The propositions (1)–(62) have been removed.
[2] The propositions (65)–(104) have been removed.
[3] The propositions (107) and (108) have been removed.
[4] The propositions (113)–(116) have been removed.

(118) Suppose $b > 0$. Then
(i) if $\frac{a}{b} \geq 1$, then $a \geq b$,
(ii) if $\frac{a}{b} \leq 1$, then $a \leq b$,
(iii) if $\frac{a}{b} \geq -1$, then $a \geq -b$ and $b \geq -a$, and
(iv) if $\frac{a}{b} \leq -1$, then $a \leq -b$ and $b \leq -a$.

(119) Suppose $b < 0$. Then
(i) if $\frac{a}{b} > 1$, then $a < b$,
(ii) if $\frac{a}{b} < 1$, then $a > b$,
(iii) if $\frac{a}{b} > -1$, then $a < -b$ and $b < -a$, and
(iv) if $\frac{a}{b} < -1$, then $a > -b$ and $b > -a$.

(120) Suppose $b < 0$. Then
(i) if $\frac{a}{b} \geq 1$, then $a \leq b$,
(ii) if $\frac{a}{b} \leq 1$, then $a \geq b$,
(iii) if $\frac{a}{b} \geq -1$, then $a \leq -b$ and $b \leq -a$, and
(iv) if $\frac{a}{b} \leq -1$, then $a \geq -b$ and $b \geq -a$.

(121) If $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$, then $a \cdot b \geq 0$.

(122) If $a < 0$ and $b < 0$ or $a > 0$ and $b > 0$, then $a \cdot b > 0$.

(123) If $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$, then $a \cdot b \leq 0$.

(125)[5] If $a \leq 0$ and $b \leq 0$ or $a \geq 0$ and $b \geq 0$, then $\frac{a}{b} \geq 0$.

(126) If $a \geq 0$ and $b < 0$ or $a \leq 0$ and $b > 0$, then $\frac{a}{b} \leq 0$.

(127) If $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$.

(128) If $a < 0$ and $b > 0$, then $\frac{a}{b} < 0$ and $\frac{b}{a} < 0$.

(129) If $a \cdot b \leq 0$, then $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$.

(132)[6] If $a \cdot b < 0$, then $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

(133) If $b \neq 0$ and $\frac{a}{b} \leq 0$, then $b > 0$ and $a \leq 0$ or $b < 0$ and $a \geq 0$.

(134) If $b \neq 0$ and $\frac{a}{b} \geq 0$, then $b > 0$ and $a \geq 0$ or $b < 0$ and $a \leq 0$.

(135) If $b \neq 0$ and $\frac{a}{b} < 0$, then $b < 0$ and $a > 0$ or $b > 0$ and $a < 0$.

(136) If $b \neq 0$ and $\frac{a}{b} > 0$, then $b > 0$ and $a > 0$ or $b < 0$ and $a < 0$.

(137) If $a > 1$ but $b > 1$ or $b \geq 1$ or $a < -1$ but $b < -1$ or $b \leq -1$, then $a \cdot b > 1$.

(138) If $a \geq 1$ and $b \geq 1$ or $a \leq -1$ and $b \leq -1$, then $a \cdot b \geq 1$.

(139) If $0 \leq a$ and $a < 1$ and $0 \leq b$ and $b \leq 1$ or $0 \geq a$ and $a > -1$ and $0 \geq b$ and $b \geq -1$, then $a \cdot b < 1$.

(140) If $0 \leq a$ and $a \leq 1$ and $0 \leq b$ and $b \leq 1$ or $0 \geq a$ and $a \geq -1$ and $0 \geq b$ and $b \geq -1$, then $a \cdot b \leq 1$.

(142)[7] If $0 < a$ and $a < b$ or $b < a$ and $a < 0$, then $\frac{a}{b} < 1$ and $\frac{b}{a} > 1$.

---
[5] The proposition (124) has been removed.
[6] The propositions (130) and (131) have been removed.
[7] The proposition (141) has been removed.

(143) If $0 < a$ and $a \leq b$ or $b \leq a$ and $a < 0$, then $\frac{a}{b} \leq 1$ and $\frac{b}{a} \geq 1$.

(144) If $a > 0$ and $b > 1$ or $a < 0$ and $b < 1$, then $a \cdot b > a$.

(145) If $a > 0$ and $b < 1$ or $a < 0$ and $b > 1$, then $a \cdot b < a$.

(146) If $a \geq 0$ and $b \geq 1$ or $a \leq 0$ and $b \leq 1$, then $a \cdot b \geq a$.

(147) If $a \geq 0$ and $b \leq 1$ or $a \leq 0$ and $b \geq 1$, then $a \cdot b \leq a$.

(149)[8] If $a < 0$, then $\frac{1}{a} < 0$ and $a^{-1} < 0$.

(150) If $\frac{1}{a} < 0$, then $a < 0$ and if $\frac{1}{a} > 0$, then $a > 0$.

(151) If $0 < a$ or $b < 0$ and if $a < b$, then $\frac{1}{a} > \frac{1}{b}$.

(152) If $0 < a$ or $b < 0$ and if $a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.

(153) If $a < 0$ and $b > 0$, then $\frac{1}{a} < \frac{1}{b}$.

(154) If $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and if $\frac{1}{a} > \frac{1}{b}$, then $a < b$.

(155) If $\frac{1}{b} > 0$ or $\frac{1}{a} < 0$ and if $\frac{1}{a} \geq \frac{1}{b}$, then $a \leq b$.

(156) If $\frac{1}{a} < 0$ and $\frac{1}{b} > 0$, then $a < b$.

(157) If $a < -1$, then $\frac{1}{a} > -1$.

(158) If $a \leq -1$, then $\frac{1}{a} \geq -1$.

(164)[9] If $1 \leq a$, then $\frac{1}{a} \leq 1$.

(165) If $b \leq e - a$, then $a \leq e - b$ and if $b \geq e - a$, then $a \geq e - b$.

(167)[10] If $a + b \leq e + d$, then $a - e \leq d - b$.

(168) If $a + b < e + d$, then $a - e < d - b$.

(169) If $a - b \leq e - d$, then $a + d \leq e + b$ and $a - e \leq b - d$ and $e - a \geq d - b$ and $b - a \geq d - e$.

(170) If $a - b < e - d$, then $a + d < e + b$ and $a - e < b - d$ and $e - a > d - b$ and $b - a > d - e$.

(171) If $a + b \leq e - d$, then $a + d \leq e - b$ and if $a + b \geq e - d$, then $a + d \geq e - b$.

(173)[11] If $a < 0$, then $a + b < b$ and $b - a > b$ and if $a + b < b$ or $b - a > b$, then $a < 0$.

(174) If $a \leq 0$, then $a + b \leq b$ and $b - a \geq b$ and if $a + b \leq b$ or $b - a \geq b$, then $a \leq 0$.

(177)[12](i) If $b > 0$ and $a \cdot b \leq e$, then $a \leq \frac{e}{b}$,
(ii) if $b < 0$ and $a \cdot b \leq e$, then $a \geq \frac{e}{b}$,
(iii) if $b > 0$ and $a \cdot b \geq e$, then $a \geq \frac{e}{b}$, and
(iv) if $b < 0$ and $a \cdot b \geq e$, then $a \leq \frac{e}{b}$.

(178)(i) If $b > 0$ and $a \cdot b < e$, then $a < \frac{e}{b}$,
(ii) if $b < 0$ and $a \cdot b < e$, then $a > \frac{e}{b}$,
(iii) if $b > 0$ and $a \cdot b > e$, then $a > \frac{e}{b}$, and
(iv) if $b < 0$ and $a \cdot b > e$, then $a < \frac{e}{b}$.

---

[8] The proposition (148) has been removed.
[9] The propositions (159)–(163) have been removed.
[10] The proposition (166) has been removed.
[11] The proposition (172) has been removed.
[12] The propositions (175) and (176) have been removed.

(181)[13] If for every a such that $a > 0$ holds $b + a \geq e$ or for every a such that $a < 0$ holds $b - a \geq e$, then $b \geq e$.

(182) If for every a such that $a > 0$ holds $b - a \leq e$ or for every a such that $a < 0$ holds $b + a \leq e$, then $b \leq e$.

(183) If for every a such that $a > 1$ holds $b \cdot a \geq e$ or for every a such that $0 < a$ and $a < 1$ holds $\frac{b}{a} \geq e$, then $b \geq e$.

(184) If for every a such that $0 < a$ and $a < 1$ holds $b \cdot a \leq e$ or for every a such that $a > 1$ holds $\frac{b}{a} \leq e$, then $b \leq e$.

(185) If $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ and if $a \cdot d < e \cdot b$, then $\frac{a}{b} < \frac{e}{d}$.

(186) If $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ and if $a \cdot d < e \cdot b$, then $\frac{a}{b} > \frac{e}{d}$.

(187) If $b > 0$ and $d > 0$ or $b < 0$ and $d < 0$ and if $a \cdot d \leq e \cdot b$, then $\frac{a}{b} \leq \frac{e}{d}$.

(188) If $b > 0$ and $d < 0$ or $b < 0$ and $d > 0$ and if $a \cdot d \leq e \cdot b$, then $\frac{a}{b} \geq \frac{e}{d}$.

(193)[14] If $b < 0$ and $d < 0$ or $b > 0$ and $d > 0$, then if $a \cdot b < \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$ and if $a \cdot b > \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$.

(194) If $b < 0$ and $d > 0$ or $b > 0$ and $d < 0$, then if $a \cdot b < \frac{e}{d}$, then $a \cdot d > \frac{e}{b}$ and if $a \cdot b > \frac{e}{d}$, then $a \cdot d < \frac{e}{b}$.

(197)[15] If $0 < a$ or $0 \leq a$ and $a < b$ or $a \leq b$ and $0 < e$ or $0 \leq e$ and $e \leq d$, then $a \cdot e \leq b \cdot d$.

(198) If $0 \geq a$ and $a \geq b$ and $0 \geq e$ and $e \geq d$, then $a \cdot e \leq b \cdot d$.

(199) If $0 < a$ and $a \leq b$ and $0 < e$ and $e < d$ or $0 > a$ and $a \geq b$ and $0 > e$ and $e > d$, then $a \cdot e < b \cdot d$.

(200) If $e > 0$ and $a > 0$ and $a < b$, then $\frac{e}{a} > \frac{e}{b}$ and if $e > 0$ and $b < 0$ and $a < b$, then $\frac{e}{a} > \frac{e}{b}$.

(201) If $e \geq 0$ and if $a > 0$ or $b < 0$ and if $a \leq b$, then $\frac{e}{a} \geq \frac{e}{b}$.

(202) If $e < 0$ and if $a > 0$ or $b < 0$ and if $a < b$, then $\frac{e}{a} < \frac{e}{b}$.

(203) If $e \leq 0$ and if $a > 0$ or $b < 0$ and if $a \leq b$, then $\frac{e}{a} \leq \frac{e}{b}$.

(204) Let X, Y be subsets of $\mathbb{R}$. Suppose $X \neq \emptyset$ and $Y \neq \emptyset$ and for all a, b such that $a \in X$ and $b \in Y$ holds $a \leq b$. Then there exists d such that for every a such that $a \in X$ holds $a \leq d$ and for every b such that $b \in Y$ holds $d \leq b$.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[3] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[13] The propositions (179) and (180) have been removed.
[14] The propositions (189)–(192) have been removed.
[15] The propositions (195) and (196) have been removed.

[4] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

Received September 5, 1990

Published January 2, 2004