JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
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Basic Properties of Rational Numbers

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Summary. A definition of rational numbers and some basic properties of them. Operations of addition, subtraction, multiplication are redefined for rational numbers. Functors numerator (num p) and denominator (den p) (p is rational) are defined and some properties of them are presented. Density of rational numbers is also given.

MML Identifier: RAT_1.

WWW: http://mizar.org/JFM/Vol2/rat_1.html

The articles [7], [4], [10], [2], [3], [8], [5], [1], [6], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: x denotes a set, a, b denote real numbers, k, k_1, l denote natural numbers, and m, m_1, n denote integers.

Let i be an integer number. Observe that $|i|$ is natural.

Let us consider k. Then $|k|$ is a natural number.

$\mathbb{Q}$ can be characterized by the condition:

(Def. 1) $x \in \mathbb{Q}$ iff there exist m, n such that $x = \frac{m}{n}$.

Let r be a number. We say that r is rational if and only if:

(Def. 2) $r \in \mathbb{Q}$.

Let us note that there exists a real number which is rational.

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A rational number is a rational number.

We now state three propositions:

(1) If $x \in \mathbb{Q}$, then there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

(3)[1] If x is a rational number, then there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

(4) $\mathbb{Q} \subseteq \mathbb{R}$.

One can check that every number which is rational is also real.

One can prove the following propositions:

(6)[2] If there exist m, n such that $x = \frac{m}{n}$, then x is rational.

(7) Every integer is a rational number.

One can check that every number which is integer is also rational.

Next we state two propositions:

[1] The proposition (2) has been removed.
[2] The proposition (5) has been removed.

(11)[3] $\mathbb{Z} \subseteq \mathbb{Q}$.

(12) $\mathbb{N} \subseteq \mathbb{Q}$.

In the sequel p, q denote rational numbers.
Let us consider p, q. One can verify the following observations:

* $p \cdot q$ is rational,
* $p + q$ is rational, and
* $p - q$ is rational.

Let us consider p, m. One can check the following observations:

* $p + m$ is rational,
* $p - m$ is rational, and
* $p \cdot m$ is rational.

Let us consider m, p. One can check the following observations:

* $m + p$ is rational,
* $m - p$ is rational, and
* $m \cdot p$ is rational.

Let us consider p, k. One can verify the following observations:

* $p + k$ is rational,
* $p - k$ is rational, and
* $p \cdot k$ is rational.

Let us consider k, p. One can verify the following observations:

* $k + p$ is rational,
* $k - p$ is rational, and
* $k \cdot p$ is rational.

Let us consider p. Note that $-p$ is rational and $|p|$ is rational.
Next we state the proposition

(16)[4] For all p, q holds $\frac{p}{q}$ is a rational number.

Let p, q be rational numbers. Observe that $\frac{p}{q}$ is rational.
The following proposition is true

(21)[5] p^{-1} is a rational number.

Let p be a rational number. One can verify that p^{-1} is rational.
Next we state three propositions:

(22) If $a < b$, then there exists p such that $a < p$ and $p < b$.

(24)[6] There exist m, k such that $k \neq 0$ and $p = \frac{m}{k}$.

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[3] The propositions (8)–(10) have been removed.
[4] The propositions (13)–(15) have been removed.
[5] The propositions (17)–(20) have been removed.
[6] The proposition (23) has been removed.

(25) There exist m, k such that $k \neq 0$ and $p = \frac{m}{k}$ and for all n, l such that $l \neq 0$ and $p = \frac{n}{l}$ holds $k \leq l$.

Let us consider p. The functor $\mathrm{den}\, p$ yields a natural number and is defined as follows:

(Def. 3) $\mathrm{den}\, p \neq 0$ and there exists m such that $p = \frac{m}{\mathrm{den}\, p}$ and for all n, k such that $k \neq 0$ and $p = \frac{n}{k}$ holds $\mathrm{den}\, p \leq k$.

Let us consider p. The functor $\mathrm{num}\, p$ yielding an integer is defined as follows:

(Def. 4) $\mathrm{num}\, p = \mathrm{den}\, p \cdot p$.

The following propositions are true:

(27)[7] $0 < \mathrm{den}\, p$.

(29)[8] $1 \leq \mathrm{den}\, p$.

(30) $0 < (\mathrm{den}\, p)^{-1}$.

(34)[9] $1 \geq (\mathrm{den}\, p)^{-1}$.

(36)[10] $\mathrm{num}\, p = 0$ iff $p = 0$.

(37) $p = \frac{\mathrm{num}\, p}{\mathrm{den}\, p}$ and $p = \mathrm{num}\, p \cdot (\mathrm{den}\, p)^{-1}$ and $p = (\mathrm{den}\, p)^{-1} \cdot \mathrm{num}\, p$.

(38) If $p \neq 0$, then $\mathrm{den}\, p = \frac{\mathrm{num}\, p}{p}$.

(40)[11] If p is an integer, then $\mathrm{den}\, p = 1$ and $\mathrm{num}\, p = p$.

(41) If $\mathrm{num}\, p = p$ or $\mathrm{den}\, p = 1$, then p is an integer.

(42) $\mathrm{num}\, p = p$ iff $\mathrm{den}\, p = 1$.

(44)[12] If $\mathrm{num}\, p = p$ or $\mathrm{den}\, p = 1$ and if $0 \leq p$, then p is a natural number.

(45) $1 < \mathrm{den}\, p$ iff p is not integer.

(46) $1 > (\mathrm{den}\, p)^{-1}$ iff p is not integer.

(47) $\mathrm{num}\, p = \mathrm{den}\, p$ iff $p = 1$.

(48) $\mathrm{num}\, p = -\mathrm{den}\, p$ iff $p = -1$.

(49) $-\mathrm{num}\, p = \mathrm{den}\, p$ iff $p = -1$.

(50) If $m \neq 0$, then $p = \frac{\mathrm{num}\, p \cdot m}{\mathrm{den}\, p \cdot m}$.

(60)[13] If $k \neq 0$ and $p = \frac{m}{k}$, then there exists l such that $m = \mathrm{num}\, p \cdot l$ and $k = \mathrm{den}\, p \cdot l$.

(61) If $p = \frac{m}{n}$ and $n \neq 0$, then there exists m_1 such that $m = \mathrm{num}\, p \cdot m_1$ and $n = \mathrm{den}\, p \cdot m_1$.

(62) It is not true that there exists l such that $1 < l$ and there exist m, k such that $\mathrm{num}\, p = m \cdot l$ and $\mathrm{den}\, p = k \cdot l$.

(63) If $p = \frac{m}{k}$ and $k \neq 0$ and it is not true that there exists l such that $1 < l$ and there exist m_1, k_1 such that $m = m_1 \cdot l$ and $k = k_1 \cdot l$, then $k = \mathrm{den}\, p$ and $m = \mathrm{num}\, p$.

[7] The proposition (26) has been removed.
[8] The proposition (28) has been removed.
[9] The propositions (31)–(33) have been removed.
[10] The proposition (35) has been removed.
[11] The proposition (39) has been removed.
[12] The proposition (43) has been removed.
[13] The propositions (51)–(59) have been removed.

(64) $p < -1$ iff $\mathrm{num}\, p < -\mathrm{den}\, p$.

(65) $p \leq -1$ iff $\mathrm{num}\, p \leq -\mathrm{den}\, p$.

(66) $p < -1$ iff $\mathrm{den}\, p < -\mathrm{num}\, p$.

(67) $p \leq -1$ iff $\mathrm{den}\, p \leq -\mathrm{num}\, p$.

(72)[14] $p < 1$ iff $\mathrm{num}\, p < \mathrm{den}\, p$.

(73) $p \leq 1$ iff $\mathrm{num}\, p \leq \mathrm{den}\, p$.

(76)[15] $p < 0$ iff $\mathrm{num}\, p < 0$.

(77) $p \leq 0$ iff $\mathrm{num}\, p \leq 0$.

(80)[16] $a < p$ iff $a \cdot \mathrm{den}\, p < \mathrm{num}\, p$.

(81) $a \leq p$ iff $a \cdot \mathrm{den}\, p \leq \mathrm{num}\, p$.

(84)[17] $p = q$ iff $\mathrm{den}\, p = \mathrm{den}\, q$ and $\mathrm{num}\, p = \mathrm{num}\, q$.

(86)[18] $p < q$ iff $\mathrm{num}\, p \cdot \mathrm{den}\, q < \mathrm{num}\, q \cdot \mathrm{den}\, p$.

(87) $\mathrm{den}(-p) = \mathrm{den}\, p$ and $\mathrm{num}(-p) = -\mathrm{num}\, p$.

(88) $0 < p$ and $q = \frac{1}{p}$ iff $\mathrm{num}\, q = \mathrm{den}\, p$ and $\mathrm{den}\, q = \mathrm{num}\, p$.

(89) $p < 0$ and $q = \frac{1}{p}$ iff $\mathrm{num}\, q = -\mathrm{den}\, p$ and $\mathrm{den}\, q = -\mathrm{num}\, p$.

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[14] The propositions (68)–(71) have been removed.
[15] The propositions (74) and (75) have been removed.
[16] The propositions (78) and (79) have been removed.
[17] The propositions (82) and (83) have been removed.
[18] The proposition (85) has been removed.

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Received July 10, 1990

Published January 2, 2004
