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σ-Fields and Probability

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Summary. This article contains definitions and theorems concerning basic properties of following objects: - a field of subsets of given nonempty set; - a sequence of subsets of given nonempty set; - a σ-field of subsets of given nonempty set and events from this σ-field; - a probability i.e. σ-additive normed measure defined on previously introduced σ-field; - a σ-field generated by family of subsets of given set; - family of Borel Sets.

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The articles [8], [4], [11], [10], [12], [2], [3], [1], [9], [6], [5], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: O_1 denotes a non empty set, X, Y, Z, p, x, y, z denote sets, D denotes a subset of O_1, f denotes a function, m, n denote natural numbers, r, r_2 denote real numbers, and s_1 denotes a sequence of real numbers.

The following two propositions are true:

(2)[1] For all r, r_2 such that $0 \leq r$ holds $r_2 - r \leq r_2$.

(3) For all r, s_1 such that there exists n such that for every m such that $n \leq m$ holds $s_1(m) = r$ holds s_1 is convergent and $\lim s_1 = r$.

Let X be a set and let I_1 be a family of subsets of X. We say that I_1 is closed for complement operator if and only if:

(Def. 1) For every subset A of X such that $A \in I_1$ holds $A^{\rm c} \in I_1$.

Let X be a set. One can verify that there exists a family of subsets of X which is non empty, closed for complement operator, and $\cap$-closed.

Let X be a set. A field of subsets of X is a non empty closed for complement operator $\cap$-closed family of subsets of X.

In the sequel F is a field of subsets of X.

One can prove the following propositions:

(4) For all subsets A, B of X holds $\{A, B\}$ is a family of subsets of X.

(6)[2] There exists a subset A of X such that $A \in F$.

(9)[3] For all sets A, B such that $A \in F$ and $B \in F$ holds $A \cup B \in F$.

[1] The proposition (1) has been removed.
[2] The proposition (5) has been removed.
[3] The propositions (7) and (8) have been removed.

(10) $\emptyset \in F$.

(11) $X \in F$.

(12) For all subsets A, B of X such that $A \in F$ and $B \in F$ holds $A \setminus B \in F$.

(13) For all sets A, B holds $A \setminus B$ misses B and if $A \in F$ and $B \in F$, then $(A \setminus B) \cup B \in F$.

(14) $\{\emptyset, X\}$ is a field of subsets of X.

(15) 2^X is a field of subsets of X.

(16) $\{\emptyset, X\} \subseteq F$ and $F \subseteq 2^X$.

(18)[4] For every p such that $p \in [:\mathbb{N}, \{X\}:]$ there exist x, y such that $\langle x, y\rangle = p$ and for all x, y, z such that $\langle x, y\rangle \in [:\mathbb{N}, \{X\}:]$ and $\langle x, z\rangle \in [:\mathbb{N}, \{X\}:]$ holds $y = z$.

(19) There exists f such that $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n) = X$.

Let X be a set. A sequence of subsets of X is a function from $\mathbb{N}$ into 2^X.
In the sequel A_1 denotes a sequence of subsets of O_1 and A_2 denotes a sequence of subsets of X.
The following two propositions are true:

(21)[5] There exists A_2 such that for every n holds $A_2(n) = X$.

(22) For all subsets A, B of X there exists A_2 such that $A_2(0) = A$ and for every n such that $n \neq 0$ holds $A_2(n) = B$.

Let us consider X, A_2, n. Then $A_2(n)$ is a subset of X.
The following proposition is true

(23) $\bigcup \operatorname{rng} A_2$ is a subset of X.

Let f be a function. The functor $\bigcup f$ yielding a set is defined as follows:

(Def. 3)[6] $\bigcup f = \bigcup \operatorname{rng} f$.

Let X be a set and let A_2 be a sequence of subsets of X. Then $\bigcup A_2$ is a subset of X.
The following two propositions are true:

(25)[7] $x \in \bigcup A_2$ iff there exists n such that $x \in A_2(n)$.

(26) There exists a sequence B_1 of subsets of X such that for every n holds $B_1(n) = A_2(n)^{\mathrm{c}}$.

Let X be a set and let A_2 be a sequence of subsets of X. The functor ComplementA_2 yielding a sequence of subsets of X is defined by:

(Def. 4) For every n holds $(\mathrm{Complement} A_2)(n) = A_2(n)^{\mathrm{c}}$.

Let X be a set and let A_2 be a sequence of subsets of X. The functor IntersectionA_2 yields a subset of X and is defined as follows:

(Def. 5) $\mathrm{Intersection} A_2 = (\bigcup \mathrm{Complement} A_2)^{\mathrm{c}}$.

One can prove the following three propositions:

(29)[8] $x \in \mathrm{Intersection} A_2$ iff for every n holds $x \in A_2(n)$.

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[4] The proposition (17) has been removed.
[5] The proposition (20) has been removed.
[6] The definition (Def. 2) has been removed.
[7] The proposition (24) has been removed.
[8] The propositions (27) and (28) have been removed.

(30) For all subsets A, B of X such that $A_2(0) = A$ and for every n such that $n \neq 0$ holds $A_2(n) = B$ holds Intersection$A_2 = A \cap B$.

(31) Complement Complement$A_2 = A_2$.

Let us consider X, A_2. We say that A_2 is non-increasing if and only if:

(Def. 6) For all n, m such that $n \leq m$ holds $A_2(m) \subseteq A_2(n)$.

We say that A_2 is non-decreasing if and only if:

(Def. 7) For all n, m such that $n \leq m$ holds $A_2(n) \subseteq A_2(m)$.

Let X be a set. A non empty family of subsets of X is said to be a σ-field of subsets of X if it satisfies the conditions (Def. 8).

(Def. 8)(i) For every sequence A_2 of subsets of X such that for every n holds $A_2(n) \in$ it holds Intersection$A_2 \in$ it, and

(ii) for every subset A of X such that $A \in$ it holds $A^{\rm c} \in$ it.

One can prove the following two propositions:

(32) Let S be a non empty set. Then S is a σ-field of subsets of X if and only if the following conditions are satisfied:

(i) $S \subseteq 2^X$,

(ii) for every sequence A_2 of subsets of X such that for every n holds $A_2(n) \in S$ holds Intersection$A_2 \in S$, and

(iii) for every subset A of X such that $A \in S$ holds $A^{\rm c} \in S$.

(35)[9] If Y is a σ-field of subsets of X, then Y is a field of subsets of X.

Let X be a set. Note that every σ-field of subsets of X is $\cap$-closed and closed for complement operator.

In the sequel S_1 is a σ-field of subsets of O_1 and S_2 is a σ-field of subsets of X.

We now state several propositions:

(38)[10] There exists a subset A of X such that $A \in S_2$.

(41)[11] For all subsets A, B of X such that $A \in S_2$ and $B \in S_2$ holds $A \cup B \in S_2$.

(42) $\emptyset \in S_2$.

(43) $X \in S_2$.

(44) For all subsets A, B of X such that $A \in S_2$ and $B \in S_2$ holds $A \setminus B \in S_2$.

Let X be a set and let S_2 be a σ-field of subsets of X. A sequence of subsets of X is said to be a sequence of subsets of S_2 if:

(Def. 9) For every n holds it$(n) \in S_2$.

Next we state the proposition

(46)[12] For every sequence A_1 of subsets of S_2 holds $\bigcup A_1 \in S_2$.

Let X be a set and let F be a σ-field of subsets of X. A subset of X is called an event of F if:

(Def. 10) It $\in F$.

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[9] The propositions (33) and (34) have been removed.
[10] The propositions (36) and (37) have been removed.
[11] The propositions (39) and (40) have been removed.
[12] The proposition (45) has been removed.

Next we state several propositions:

(48)[13] If $x \in S_2$, then x is an event of S_2.

(49) For all events A, B of S_2 holds $A \cap B$ is an event of S_2.

(50) For every event A of S_2 holds A^{c} is an event of S_2.

(51) For all events A, B of S_2 holds $A \cup B$ is an event of S_2.

(52) $\emptyset$ is an event of S_2.

(53) X is an event of S_2.

(54) For all events A, B of S_2 holds $A \setminus B$ is an event of S_2.

Let us consider X, S_2. One can verify that there exists an event of S_2 which is empty.
Let us consider X, S_2. The functor $\Omega_{(S_2)}$ yields an event of S_2 and is defined as follows:

(Def. 11) $\Omega_{(S_2)} = X$.

Let us consider X, S_2 and let A, B be events of S_2. Then $A \cap B$ is an event of S_2. Then $A \cup B$ is an event of S_2. Then $A \setminus B$ is an event of S_2.
The following two propositions are true:

(57)[14] A_1 is a sequence of subsets of S_1 iff for every n holds $A_1(n)$ is an event of S_1.

(58) If A_1 is a sequence of subsets of S_1, then $\bigcup A_1$ is an event of S_1.

In the sequel A, B are events of S_1 and A_1 is a sequence of subsets of S_1.
Next we state the proposition

(59) There exists f such that $\mathrm{dom}\, f = S_1$ and for every D such that $D \in S_1$ holds if $p \in D$, then $f(D) = 1$ and if $p \notin D$, then $f(D) = 0$.

In the sequel P denotes a function from S_1 into $\mathbb{R}$.
The following two propositions are true:

(60) There exists P such that for every D such that $D \in S_1$ holds if $p \in D$, then $P(D) = 1$ and if $p \notin D$, then $P(D) = 0$.

(62)[15] $P \cdot A_1$ is a sequence of real numbers.

Let us consider O_1, S_1, A_1, P. Then $P \cdot A_1$ is a sequence of real numbers.
Let us consider O_1, S_1, P, A. Then $P(A)$ is a real number.
Let us consider O_1, S_1. A function from S_1 into $\mathbb{R}$ is said to be a probability on S_1 if it satisfies the conditions (Def. 13).

(Def. 13)[16](i) For every A holds $0 \leq \mathrm{it}(A)$,

(ii) $\mathrm{it}(O_1) = 1$,

(iii) for all A, B such that A misses B holds $\mathrm{it}(A \cup B) = \mathrm{it}(A) + \mathrm{it}(B)$, and

(iv) for every A_1 such that A_1 is non-increasing holds $\mathrm{it} \cdot A_1$ is convergent and $\lim(\mathrm{it} \cdot A_1) = \mathrm{it}(\mathrm{Intersection} A_1)$.

In the sequel P denotes a probability on S_1.
Next we state a number of propositions:

[13] The proposition (47) has been removed.
[14] The propositions (55) and (56) have been removed.
[15] The proposition (61) has been removed.
[16] The definition (Def. 12) has been removed.

(64)[17] $P(\emptyset) = 0$.

(66)[18] $P(\Omega_{(S_1)}) = 1$.

(67) $P(\Omega_{(S_1)} \setminus A) + P(A) = 1$.

(68) $P(\Omega_{(S_1)} \setminus A) = 1 - P(A)$.

(69) If $A \subseteq B$, then $P(B \setminus A) = P(B) - P(A)$.

(70) If $A \subseteq B$, then $P(A) \leq P(B)$.

(71) $P(A) \leq 1$.

(72) $P(A \cup B) = P(A) + P(B \setminus A)$.

(73) $P(A \cup B) = P(A) + P(B \setminus A \cap B)$.

(74) $P(A \cup B) = (P(A) + P(B)) - P(A \cap B)$.

(75) $P(A \cup B) \leq P(A) + P(B)$.

In the sequel D denotes a subset of $\mathbb{R}$.
We now state the proposition

(76) 2^{O_1} is a σ-field of subsets of O_1.

Let us consider O_1 and let X be a subset of 2^{O_1}. The functor $\sigma(X)$ yielding a σ-field of subsets of O_1 is defined by:

(Def. 14) $X \subseteq \sigma(X)$ and for every Z such that $X \subseteq Z$ and Z is a σ-field of subsets of O_1 holds $\sigma(X) \subseteq Z$.

Let us consider r. The functor $\mathrm{HL}(r)$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 15) $\mathrm{HL}(r) = \{r_1; r_1$ ranges over elements of $\mathbb{R}$: $r_1 < r\}$.

The subset Halflines of $2^{\mathbb{R}}$ is defined as follows:

(Def. 16) Halflines $= \{D : \bigvee_r D = \mathrm{HL}(r)\}$.

The σ-field the Borel sets of subsets of $\mathbb{R}$ is defined by:

(Def. 17) The Borel sets $= \sigma(\text{Halflines})$.

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[17] The proposition (63) has been removed.
[18] The proposition (65) has been removed.

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