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# Many-sorted Sets

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**Summary.** The article deals with parameterized families of sets. When treated in a similar way as sets (due to systematic overloading notation used for sets) they are called many sorted sets. For instance, if $x$ and $X$ are two many-sorted sets (with the same set of indices $I$) then relation $x \in X$ is defined as $\forall_{i \in I} x_i \in X_i$.

I was prompted by a remark in a paper by Tarlecki and Wirsing: "Throughout the paper we deal with many-sorted sets, functions, relations etc. ... We feel free to use any standard set-theoretic notation without explicit use of indices" [6, p. 97]. The aim of this work was to check the feasibility of such approach in Mizar. It works.

Let us observe some peculiarities:

- empty set (i.e. the many sorted set with empty set of indices) belongs to itself (theorem 133),
- we get two different inclusions $X \subseteq Y$ iff $\forall_{i \in I} X_i \subseteq Y_i$ and $X \sqsubseteq Y$ iff $\forall_x x \in X \Rightarrow x \in Y$ equivalent only for sets that yield non empty values.

Therefore the care is advised.

MML Identifier: PBOOLE.

WWW: http://mizar.org/JFM/Vol5/pboole.html

The articles [8], [9], [10], [2], [7], [3], [1], [5], and [4] provide the notation and terminology for this paper.

## 1. Preliminaries

In this paper $i$, $e$ are sets.

Let $f$ be a function. Let us observe that $f$ is empty yielding if and only if:

(Def. 1) For every $i$ such that $i \in \operatorname{dom} f$ holds $f(i)$ is empty.

Let us note that there exists a function which is empty yielding.

We now state two propositions:

(1) For every function $f$ such that $f$ is non-empty holds $\operatorname{rng} f$ has non empty elements.

(2) For every function $f$ holds $f$ is empty yielding iff $f = \emptyset$ or $\operatorname{rng} f = \{\emptyset\}$.

In the sequel $I$ denotes a set.

Let us consider $I$. A function is called a many sorted set indexed by $I$ if:

(Def. 3)[1] $\operatorname{dom} \mathrm{it} = I$.

[1] The definition (Def. 2) has been removed.

In the sequel $x$, $X$, $Y$, $Z$, $V$ are many sorted sets indexed by $I$.

The scheme *Kuratowski Function* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a many sorted set $f$ indexed by $\mathcal{A}$ such that for every $e$ such that $e \in \mathcal{A}$ holds $f(e) \in \mathcal{F}(e)$

provided the parameters satisfy the following condition:

- For every $e$ such that $e \in \mathcal{A}$ holds $\mathcal{F}(e) \neq \emptyset$.

Let us consider $I$, $X$, $Y$. The predicate $X \in Y$ is defined by:

(Def. 4) For every $i$ such that $i \in I$ holds $X(i) \in Y(i)$.

The predicate $X \subseteq Y$ is defined by:

(Def. 5) For every $i$ such that $i \in I$ holds $X(i) \subseteq Y(i)$.

Let us note that the predicate $X \subseteq Y$ is reflexive.

Let $I$ be a non empty set and let $X$, $Y$ be many sorted sets indexed by $I$. Let us note that the predicate $X \in Y$ is antisymmetric.

The scheme *PSeparation* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a many sorted set $X$ indexed by $\mathcal{A}$ such that for every set $i$ if $i \in \mathcal{A}$, then for every $e$ holds $e \in X(i)$ iff $e \in \mathcal{B}(i)$ and $\mathcal{P}[i,e]$

for all values of the parameters.

The following proposition is true

(3) If for every $i$ such that $i \in I$ holds $X(i) = Y(i)$, then $X = Y$.

Let us consider $I$. The functor $\mathbf{0}_I$ yields a many sorted set indexed by $I$ and is defined by:

(Def. 6) $\mathbf{0}_I = I \longmapsto \emptyset$.

Let us consider $X$, $Y$. The functor $X \cup Y$ yields a many sorted set indexed by $I$ and is defined as follows:

(Def. 7) For every $i$ such that $i \in I$ holds $(X \cup Y)(i) = X(i) \cup Y(i)$.

Let us observe that the functor $X \cup Y$ is commutative and idempotent. The functor $X \cap Y$ yielding a many sorted set indexed by $I$ is defined by:

(Def. 8) For every $i$ such that $i \in I$ holds $(X \cap Y)(i) = X(i) \cap Y(i)$.

Let us observe that the functor $X \cap Y$ is commutative and idempotent. The functor $X \setminus Y$ yielding a many sorted set indexed by $I$ is defined by:

(Def. 9) For every $i$ such that $i \in I$ holds $(X \setminus Y)(i) = X(i) \setminus Y(i)$.

We say that $X$ overlaps $Y$ if and only if:

(Def. 10) For every $i$ such that $i \in I$ holds $X(i)$ meets $Y(i)$.

Let us note that the predicate $X$ overlaps $Y$ is symmetric. We say that $X$ misses $Y$ if and only if:

(Def. 11) For every $i$ such that $i \in I$ holds $X(i)$ misses $Y(i)$.

Let us note that the predicate $X$ misses $Y$ is symmetric. We introduce $X$ meets $Y$ as an antonym of $X$ misses $Y$.

Let us consider $I$, $X$, $Y$. The functor $X \dot{-} Y$ yielding a many sorted set indexed by $I$ is defined as follows:

(Def. 12) $X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X)$.

Let us observe that the functor $X \dot{-} Y$ is commutative.

The following propositions are true:

(4) For every $i$ such that $i \in I$ holds $(X \dot{-} Y)(i) = X(i) \dot{-} Y(i)$.

(5) For every $i$ such that $i \in I$ holds $\mathbf{0}_I(i) = \emptyset$.

(6) If for every $i$ such that $i \in I$ holds $X(i) = \emptyset$, then $X = \mathbf{0}_I$.

(7) If $x \in X$ or $x \in Y$, then $x \in X \cup Y$.

(8) $x \in X \cap Y$ iff $x \in X$ and $x \in Y$.

(9) If $x \in X$ and $X \subseteq Y$, then $x \in Y$.

(10) If $x \in X$ and $x \in Y$, then $X$ overlaps $Y$.

(11) If $X$ overlaps $Y$, then there exists $x$ such that $x \in X$ and $x \in Y$.

(12) If $x \in X \setminus Y$, then $x \in X$.

## 2. Lattice Properties of Many Sorted Sets

We now state the proposition

(13) $X \subseteq X$.

Let us consider $I$, $X$, $Y$. Let us observe that $X = Y$ if and only if:

(Def. 13) $X \subseteq Y$ and $Y \subseteq X$.

We now state a number of propositions:

(15)[2] If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(16) $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$.

(17) $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

(18) If $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.

(19) If $Z \subseteq X$ and $Z \subseteq Y$, then $Z \subseteq X \cap Y$.

(20) If $X \subseteq Y$, then $X \cup Z \subseteq Y \cup Z$ and $Z \cup X \subseteq Z \cup Y$.

(21) If $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$ and $Z \cap X \subseteq Z \cap Y$.

(22) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cup Z \subseteq Y \cup V$.

(23) If $X \subseteq Y$ and $Z \subseteq V$, then $X \cap Z \subseteq Y \cap V$.

(24) If $X \subseteq Y$, then $X \cup Y = Y$ and $Y \cup X = Y$.

(25) If $X \subseteq Y$, then $X \cap Y = X$ and $Y \cap X = X$.

(26) $X \cap Y \subseteq X \cup Z$.

(27) If $X \subseteq Z$, then $X \cup Y \cap Z = (X \cup Y) \cap Z$.

(28) $X = Y \cup Z$ iff $Y \subseteq X$ and $Z \subseteq X$ and for every $V$ such that $Y \subseteq V$ and $Z \subseteq V$ holds $X \subseteq V$.

(29) $X = Y \cap Z$ iff $X \subseteq Y$ and $X \subseteq Z$ and for every $V$ such that $V \subseteq Y$ and $V \subseteq Z$ holds $V \subseteq X$.

(34)[3] $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

(35) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

[2] The proposition (14) has been removed.

[3] The propositions (30)–(33) have been removed.

(36) $X \cap (X \cup Y) = X$ and $(X \cup Y) \cap X = X$ and $X \cap (Y \cup X) = X$ and $(Y \cup X) \cap X = X$.

(37) $X \cup X \cap Y = X$ and $X \cap Y \cup X = X$ and $X \cup Y \cap X = X$ and $Y \cap X \cup X = X$.

(38) $X \cap (Y \cup Z) = X \cap Y \cup X \cap Z$.

(39) $X \cup Y \cap Z = (X \cup Y) \cap (X \cup Z)$ and $Y \cap Z \cup X = (Y \cup X) \cap (Z \cup X)$.

(40) If $X \cap Y \cup X \cap Z = X$, then $X \subseteq Y \cup Z$.

(41) If $(X \cup Y) \cap (X \cup Z) = X$, then $Y \cap Z \subseteq X$.

(42) $X \cap Y \cup Y \cap Z \cup Z \cap X = (X \cup Y) \cap (Y \cup Z) \cap (Z \cup X)$.

(43) If $X \cup Y \subseteq Z$, then $X \subseteq Z$ and $Y \subseteq Z$.

(44) If $X \subseteq Y \cap Z$, then $X \subseteq Y$ and $X \subseteq Z$.

(45) $(X \cup Y) \cup Z = X \cup Z \cup (Y \cup Z)$ and $X \cup (Y \cup Z) = (X \cup Y) \cup (X \cup Z)$.

(46) $(X \cap Y) \cap Z = X \cap Z \cap (Y \cap Z)$ and $X \cap (Y \cap Z) = (X \cap Y) \cap (X \cap Z)$.

(47) $X \cup (X \cup Y) = X \cup Y$ and $X \cup Y \cup Y = X \cup Y$.

(48) $X \cap (X \cap Y) = X \cap Y$ and $X \cap Y \cap Y = X \cap Y$.

## 3. The Empty Many Sorted Set

Next we state several propositions:

(49) $\mathbf{0}_I \subseteq X$.

(50) If $X \subseteq \mathbf{0}_I$, then $X = \mathbf{0}_I$.

(51) If $X \subseteq Y$ and $X \subseteq Z$ and $Y \cap Z = \mathbf{0}_I$, then $X = \mathbf{0}_I$.

(52) If $X \subseteq Y$ and $Y \cap Z = \mathbf{0}_I$, then $X \cap Z = \mathbf{0}_I$.

(53) $X \cup \mathbf{0}_I = X$ and $\mathbf{0}_I \cup X = X$.

(54) If $X \cup Y = \mathbf{0}_I$, then $X = \mathbf{0}_I$ and $Y = \mathbf{0}_I$.

(55) $X \cap \mathbf{0}_I = \mathbf{0}_I$ and $\mathbf{0}_I \cap X = \mathbf{0}_I$.

(56) If $X \subseteq Y \cup Z$ and $X \cap Z = \mathbf{0}_I$, then $X \subseteq Y$.

(57) If $Y \subseteq X$ and $X \cap Y = \mathbf{0}_I$, then $Y = \mathbf{0}_I$.

## 4. The Difference and the Symmetric Difference

One can prove the following propositions:

(58) $X \setminus Y = \mathbf{0}_I$ iff $X \subseteq Y$.

(59) If $X \subseteq Y$, then $X \setminus Z \subseteq Y \setminus Z$.

(60) If $X \subseteq Y$, then $Z \setminus Y \subseteq Z \setminus X$.

(61) If $X \subseteq Y$ and $Z \subseteq V$, then $X \setminus V \subseteq Y \setminus Z$.

(62) $X \setminus Y \subseteq X$.

(63) If $X \subseteq Y \setminus X$, then $X = \mathbf{0}_I$.

(64) $X \setminus X = \mathbf{0}_I$.

(65) $X \setminus \mathbf{0}_I = X$.

(66) $\mathbf{0}_I \setminus X = \mathbf{0}_I$.

(67) $X \setminus (X \cup Y) = \mathbf{0}_I$ and $X \setminus (Y \cup X) = \mathbf{0}_I$.

(68) $X \cap (Y \setminus Z) = X \cap Y \setminus Z$.

(69) $(X \setminus Y) \cap Y = \mathbf{0}_I$ and $Y \cap (X \setminus Y) = \mathbf{0}_I$.

(70) $X \setminus (Y \setminus Z) = (X \setminus Y) \cup X \cap Z$.

(71) $(X \setminus Y) \cup X \cap Y = X$ and $X \cap Y \cup (X \setminus Y) = X$.

(72) If $X \subseteq Y$, then $Y = X \cup (Y \setminus X)$ and $Y = (Y \setminus X) \cup X$.

(73) $X \cup (Y \setminus X) = X \cup Y$ and $(Y \setminus X) \cup X = Y \cup X$.

(74) $X \setminus (X \setminus Y) = X \cap Y$.

(75) $X \setminus Y \cap Z = (X \setminus Y) \cup (X \setminus Z)$.

(76) $X \setminus X \cap Y = X \setminus Y$ and $X \setminus Y \cap X = X \setminus Y$.

(77) $X \cap Y = \mathbf{0}_I$ iff $X \setminus Y = X$.

(78) $(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z)$.

(79) $X \setminus Y \setminus Z = X \setminus (Y \cup Z)$.

(80) $X \cap Y \setminus Z = (X \setminus Z) \cap (Y \setminus Z)$.

(81) $(X \cup Y) \setminus Y = X \setminus Y$.

(82) If $X \subseteq Y \cup Z$, then $X \setminus Y \subseteq Z$ and $X \setminus Z \subseteq Y$.

(83) $(X \cup Y) \setminus X \cap Y = (X \setminus Y) \cup (Y \setminus X)$.

(84) $X \setminus Y \setminus Y = X \setminus Y$.

(85) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$.

(86) If $X \setminus Y = Y \setminus X$, then $X = Y$.

(87) $X \cap (Y \setminus Z) = X \cap Y \setminus X \cap Z$.

(88) If $X \setminus Y \subseteq Z$, then $X \subseteq Y \cup Z$.

(89) $X \setminus Y \subseteq X \dot{-} Y$.

(91)[4] $X \dot{-} \mathbf{0}_I = X$ and $\mathbf{0}_I \dot{-} X = X$.

(92) $X \dot{-} X = \mathbf{0}_I$.

(94)[5] $X \cup Y = (X \dot{-} Y) \cup X \cap Y$.

(95) $X \dot{-} Y = (X \cup Y) \setminus X \cap Y$.

(96) $(X \dot{-} Y) \setminus Z = (X \setminus (Y \cup Z)) \cup (Y \setminus (X \cup Z))$.

(97) $X \setminus (Y \dot{-} Z) = (X \setminus (Y \cup Z)) \cup X \cap Y \cap Z$.

(98) $(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z)$.

(99) If $X \setminus Y \subseteq Z$ and $Y \setminus X \subseteq Z$, then $X \dot{-} Y \subseteq Z$.

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[4] The proposition (90) has been removed.

[5] The proposition (93) has been removed.

(100) $X \cup Y = X \dot{-} (Y \setminus X)$.

(101) $X \cap Y = X \dot{-} (X \setminus Y)$.

(102) $X \setminus Y = X \dot{-} X \cap Y$.

(103) $Y \setminus X = X \dot{-} (X \cup Y)$.

(104) $X \cup Y = X \dot{-} Y \dot{-} X \cap Y$.

(105) $X \cap Y = X \dot{-} Y \dot{-} (X \cup Y)$.

## 5. Meeting and Overlapping

Next we state a number of propositions:

(106) If $X$ overlaps $Y$ or $X$ overlaps $Z$, then $X$ overlaps $Y \cup Z$.

(108)[6] If $X$ overlaps $Y$ and $Y \subseteq Z$, then $X$ overlaps $Z$.

(109) If $X$ overlaps $Y$ and $X \subseteq Z$, then $Z$ overlaps $Y$.

(110) If $X \subseteq Y$ and $Z \subseteq V$ and $X$ overlaps $Z$, then $Y$ overlaps $V$.

(111) If $X$ overlaps $Y \cap Z$, then $X$ overlaps $Y$ and $X$ overlaps $Z$.

(112) If $X$ overlaps $Z$ and $X \subseteq V$, then $X$ overlaps $Z \cap V$.

(113) If $X$ overlaps $Y \setminus Z$, then $X$ overlaps $Y$.

(114) If $Y$ does not overlap $Z$, then $X \cap Y$ does not overlap $X \cap Z$.

(115) If $X$ overlaps $Y \setminus Z$, then $Y$ overlaps $X \setminus Z$.

(116) If $X$ meets $Y$ and $Y \subseteq Z$, then $X$ meets $Z$.

(118)[7] $Y$ misses $X \setminus Y$.

(119) $X \cap Y$ misses $X \setminus Y$.

(120) $X \cap Y$ misses $X \dot{-} Y$.

(121) If $X$ misses $Y$, then $X \cap Y = \mathbf{0}_I$.

(122) If $X \neq \mathbf{0}_I$, then $X$ meets $X$.

(123) If $X \subseteq Y$ and $X \subseteq Z$ and $Y$ misses $Z$, then $X = \mathbf{0}_I$.

(124) If $Z \cup V = X \cup Y$ and $X$ misses $Z$ and $Y$ misses $V$, then $X = V$ and $Y = Z$.

(126)[8] If $X$ misses $Y$, then $X \setminus Y = X$.

(127) If $X$ misses $Y$, then $(X \cup Y) \setminus Y = X$.

(128) If $X \setminus Y = X$, then $X$ misses $Y$.

(129) $X \setminus Y$ misses $Y \setminus X$.

[6] The proposition (107) has been removed.
[7] The proposition (117) has been removed.
[8] The proposition (125) has been removed.

## 6. The Second Inclusion

Let us consider $I$, $X$, $Y$. The predicate $X \sqsubseteq Y$ is defined by:

(Def. 14) For every $x$ such that $x \in X$ holds $x \in Y$.

Let us note that the predicate $X \sqsubseteq Y$ is reflexive.

The following propositions are true:

(130) If $X \subseteq Y$, then $X \sqsubseteq Y$.

(131) $X \sqsubseteq X$.

(132) If $X \sqsubseteq Y$ and $Y \sqsubseteq Z$, then $X \sqsubseteq Z$.

## 7. Non empty set of sorts

We now state two propositions:

(133) $\mathbf{0}_\emptyset \in \mathbf{0}_\emptyset$.

(134) For every many sorted set $X$ indexed by $\emptyset$ holds $X = \emptyset$.

In the sequel $I$ denotes a non empty set and $x$, $X$, $Y$ denote many sorted sets indexed by $I$.

We now state several propositions:

(135) If $X$ overlaps $Y$, then $X$ meets $Y$.

(136) It is not true that there exists $x$ such that $x \in \mathbf{0}_I$.

(137) If $x \in X$ and $x \in Y$, then $X \cap Y \neq \mathbf{0}_I$.

(138) $X$ does not overlap $\mathbf{0}_I$ and $\mathbf{0}_I$ does not overlap $X$.

(139) If $X \cap Y = \mathbf{0}_I$, then $X$ does not overlap $Y$.

(140) If $X$ overlaps $X$, then $X \neq \mathbf{0}_I$.

## 8. Non Empty and Non-empty Many Sorted Sets

We adopt the following rules: $I$ is a set and $x$, $X$, $Y$, $Z$ are many sorted sets indexed by $I$.

Let $I$ be a set and let $X$ be a many sorted set indexed by $I$. Let us observe that $X$ is empty yielding if and only if:

(Def. 15) For every $i$ such that $i \in I$ holds $X(i)$ is empty.

Let us observe that $X$ is non-empty if and only if:

(Def. 16) For every $i$ such that $i \in I$ holds $X(i)$ is non empty.

Let $I$ be a set. Note that there exists a many sorted set indexed by $I$ which is empty yielding and there exists a many sorted set indexed by $I$ which is non-empty.

Let $I$ be a non empty set. Observe that every many sorted set indexed by $I$ which is non-empty is also non empty yielding and every many sorted set indexed by $I$ which is empty yielding is also non non-empty.

The following propositions are true:

(141) $X$ is empty yielding iff $X = \mathbf{0}_I$.

(142) If $Y$ is empty yielding and $X \subseteq Y$, then $X$ is empty yielding.

(143) If $X$ is non-empty and $X \subseteq Y$, then $Y$ is non-empty.

(144) If $X$ is non-empty and $X \sqsubseteq Y$, then $X \subseteq Y$.

(145) If $X$ is non-empty and $X \sqsubseteq Y$, then $Y$ is non-empty.

In the sequel $X$ denotes a non-empty many sorted set indexed by $I$.
Next we state three propositions:

(146) There exists $x$ such that $x \in X$.

(147) If for every $x$ holds $x \in X$ iff $x \in Y$, then $X = Y$.

(148) If for every $x$ holds $x \in X$ iff $x \in Y$ and $x \in Z$, then $X = Y \cap Z$.

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