

Homomorphisms of Order Sorted Algebras¹

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The articles [8], [11], [13], [14], [4], [5], [2], [1], [10], [7], [9], [3], [12], and [6] provide the notation and terminology for this paper.

In this paper R denotes a non empty poset and S_1 denotes an order sorted signature.

Let us consider R and let F be a many sorted function indexed by the carrier of R . We say that F is order-sorted if and only if:

(Def. 1) For all elements s_1, s_2 of R such that $s_1 \leq s_2$ and for every set a_1 such that $a_1 \in \text{dom } F(s_1)$ holds $a_1 \in \text{dom } F(s_2)$ and $F(s_1)(a_1) = F(s_2)(a_1)$.

Next we state four propositions:

- (2)¹ Let F be a many sorted function indexed by the carrier of R . Suppose F is order-sorted. Let s_1, s_2 be elements of R . If $s_1 \leq s_2$, then $\text{dom } F(s_1) \subseteq \text{dom } F(s_2)$ and $F(s_1) \subseteq F(s_2)$.
- (3) Let A be an order sorted set of R , B be a non-empty order sorted set of R , and F be a many sorted function from A into B . Then F is order-sorted if and only if for all elements s_1, s_2 of R such that $s_1 \leq s_2$ and for every set a_1 such that $a_1 \in A(s_1)$ holds $F(s_1)(a_1) = F(s_2)(a_1)$.
- (4) Let F be a many sorted function indexed by the carrier of R . Suppose F is order-sorted. Let w_1, w_2 be elements of $(\text{the carrier of } R)^*$. If $w_1 \leq w_2$, then $F^\#(w_1) \subseteq F^\#(w_2)$.
- (5) For every order sorted set A of R holds id_A is order-sorted.

Let us consider R and let A be an order sorted set of R . Observe that id_A is order-sorted.

The following propositions are true:

- (6) Let A be an order sorted set of R , B, C be non-empty order sorted sets of R , F be a many sorted function from A into B , and G be a many sorted function from B into C . If F is order-sorted and G is order-sorted, then $G \circ F$ is order-sorted.
- (7) Let A, B be order sorted sets of R and F be a many sorted function from A into B . If F is “1-1”, onto, and order-sorted, then F^{-1} is order-sorted.
- (8) Let A be an order sorted set of R and F be a many sorted function indexed by the carrier of R . If F is order-sorted, then $F \circ A$ is an order sorted set of R .

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¹ The proposition (1) has been removed.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1 . We say that U_1 and U_2 are os-isomorphic if and only if:

(Def. 2) There exists a many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2 and order-sorted.

One can prove the following two propositions:

- (9) For every order sorted algebra U_1 of S_1 holds U_1 and U_1 are os-isomorphic.
- (10) Let U_1, U_2 be non-empty order sorted algebras of S_1 . If U_1 and U_2 are os-isomorphic, then U_2 and U_1 are os-isomorphic.

Let us consider S_1 and let U_1, U_2 be order sorted algebras of S_1 . Let us note that the predicate U_1 and U_2 are os-isomorphic is reflexive.

Let us consider S_1 and let U_1, U_2 be non-empty order sorted algebras of S_1 . Let us note that the predicate U_1 and U_2 are os-isomorphic is symmetric.

One can prove the following propositions:

- (11) Let U_1, U_2, U_3 be non-empty order sorted algebras of S_1 . Suppose U_1 and U_2 are os-isomorphic and U_2 and U_3 are os-isomorphic. Then U_1 and U_3 are os-isomorphic.
- (12) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2 . Suppose F is order-sorted and a homomorphism of U_1 into U_2 . Then $\text{Im} F$ is order-sorted.
- (13) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2 . Suppose F is order-sorted. Let o_1, o_2 be operation symbols of S_1 . Suppose $o_1 \leq o_2$. Let x be an element of $\text{Args}(o_1, U_1)$ and x_1 be an element of $\text{Args}(o_2, U_1)$. If $x = x_1$, then $F\#x = F\#x_1$.
- (14) Let U_1 be a monotone non-empty order sorted algebra of S_1 , U_2 be a non-empty order sorted algebra of S_1 , and F be a many sorted function from U_1 into U_2 . Suppose F is order-sorted and a homomorphism of U_1 into U_2 . Then $\text{Im} F$ is order-sorted and $\text{Im} F$ is a monotone order sorted algebra of S_1 .
- (15) For every monotone order sorted algebra U_1 of S_1 holds every OSSubAlgebra of U_1 is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1 . Observe that there exists an OSSubAlgebra of U_1 which is monotone.

Let us consider S_1 and let U_1 be a monotone order sorted algebra of S_1 . Observe that every OSSubAlgebra of U_1 is monotone.

Next we state two propositions:

- (16) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2 . Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function G from U_1 into $\text{Im} F$ such that $F = G$ and G is order-sorted and an epimorphism of U_1 onto $\text{Im} F$.
- (17) Let U_1, U_2 be non-empty order sorted algebras of S_1 and F be a many sorted function from U_1 into U_2 . Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then there exists a many sorted function F_1 from U_1 into $\text{Im} F$ and there exists a many sorted function F_2 from $\text{Im} F$ into U_2 such that
 - (i) F_1 is an epimorphism of U_1 onto $\text{Im} F$,
 - (ii) F_2 is a monomorphism of $\text{Im} F$ into U_2 ,
 - (iii) $F = F_2 \circ F_1$,
 - (iv) F_1 is order-sorted, and
 - (v) F_2 is order-sorted.

Let us consider S_1 and let U_1 be an order sorted algebra of S_1 . One can verify that \langle the sorts of U_1 , the characteristics of $U_1\rangle$ is order-sorted.

We now state two propositions:

- (18) Let U_1 be an order sorted algebra of S_1 . Then U_1 is monotone if and only if \langle the sorts of U_1 , the characteristics of $U_1\rangle$ is monotone.
- (19) Let U_1, U_2 be strict non-empty order sorted algebras of S_1 . Suppose U_1 and U_2 are os-isomorphic. Then U_1 is monotone if and only if U_2 is monotone.

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