JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Construction of a bilinear symmetric form in orthogonal vector space[1]

Eugeniusz Kusak
Warsaw University
Białystok

Wojciech Leończuk
Warsaw University
Białystok

Michał Muzalewski
Warsaw University
Białystok

Summary. In this text we present unpublished results by Eugeniusz Kusak and Wojciech Leończuk. They contain an axiomatic description of the class of all spaces $\langle V;\perp_\xi\rangle$, where V is a vector space over a field F, $\xi : V \times V \to F$ is a bilinear symmetric form i.e. $\xi(x,y) = \xi(y,x)$ and $x \perp_\xi y$ iff $\xi(x,y) = 0$ for $x, y \in V$. They also contain an effective construction of bilinear symmetric form ξ for given orthogonal space $\langle V;\perp\rangle$ such that $\perp = \perp_\xi$. The basic tool used in this method is the notion of orthogonal projection $\mathrm{J}(a,b,x)$ for $a,b,x \in V$. We should stress the fact that axioms of orthogonal and symplectic spaces differ only by one axiom, namely: $x \perp y+\varepsilon z \& y \perp z+\varepsilon x \Rightarrow z \perp x+\varepsilon y$. For $\varepsilon = -1$ we get the axiom on three perpendiculars characterizing orthogonal geometry. For $\varepsilon = +1$ we get the axiom characterizing symplectic geometry - see [5].

MML Identifier: ORTSP_1.

WWW: http://mizar.org/JFM/Vol1/ortsp_1.html

The articles [6], [3], [8], [1], [2], [7], [4], [9], and [5] provide the notation and terminology for this paper.

In this paper F is a field.

Let us consider F and let I_1 be an Abelian add-associative right zeroed right complementable non empty simplectic structure over F. We say that I_1 is orthogonality space-like if and only if the condition (Def. 2) is satisfied.

(Def. 2)[1] Let a, b, c, d, x be elements of I_1 and l be an element of F. Then

- (i) if $a \neq 0_{(I_1)}$ and $b \neq 0_{(I_1)}$ and $c \neq 0_{(I_1)}$ and $d \neq 0_{(I_1)}$, then there exists an element p of I_1 such that $p \not\perp a$ and $p \not\perp b$ and $p \not\perp c$ and $p \not\perp d$,
- (ii) if $a \perp b$, then $l \cdot a \perp b$,
- (iii) if $b \perp a$ and $c \perp a$, then $b+c \perp a$,
- (iv) if $b \not\perp a$, then there exists an element k of F such that $x - k \cdot b \perp a$, and
- (v) if $a \perp b-c$ and $b \perp c-a$, then $c \perp a-b$.

Let us consider F. Observe that there exists an Abelian add-associative right zeroed right complementable non empty simplectic structure over F which is orthogonality space-like, vector space-like, and strict.

Let us consider F. An orthogonality space over F is an orthogonality space-like vector space-like Abelian add-associative right zeroed right complementable non empty simplectic structure over F.

[1]Supported by RPBP.III-24.C6.

[1] The definition (Def. 1) has been removed.

We use the following convention: S is an orthogonality space over F, a, b, c, d, p, q, x, y, z are elements of S, and k, l are elements of F.

We now state a number of propositions:

(11)[2] $0_S \perp a$.

(12) If $a \perp b$, then $b \perp a$.

(13) If $a \not\perp b$ and $c + a \perp b$, then $c \not\perp b$.

(14) If $b \not\perp a$ and $c \perp a$, then $b + c \not\perp a$.

(15) If $b \not\perp a$ and $l \neq 0_F$, then $l \cdot b \not\perp a$ and $b \not\perp l \cdot a$.

(16) If $a \perp b$, then $-a \perp b$.

(19)[3] If $a - b \perp d$ and $a - c \perp d$, then $b - c \perp d$.

(20) If $b \not\perp a$ and $x - k \cdot b \perp a$ and $x - l \cdot b \perp a$, then $k = l$.

(21) If $a \perp a$ and $b \perp b$, then $a + b \perp a - b$.

(22) If $\mathbf{1}_F + \mathbf{1}_F \neq 0_F$ and there exists a such that $a \neq 0_S$, then there exists b such that $b \not\perp b$.

Let us consider F, S, a, b, x. Let us assume that $b \not\perp a$. The functor $\mathrm{J}(a, b, x)$ yielding an element of F is defined as follows:

(Def. 6)[4] For every element l of F such that $x - l \cdot b \perp a$ holds $\mathrm{J}(a, b, x) = l$.

The following propositions are true:

(24)[5] If $b \not\perp a$, then $x - \mathrm{J}(a, b, x) \cdot b \perp a$.

(25) If $b \not\perp a$, then $\mathrm{J}(a, b, l \cdot x) = l \cdot \mathrm{J}(a, b, x)$.

(26) If $b \not\perp a$, then $\mathrm{J}(a, b, x + y) = \mathrm{J}(a, b, x) + \mathrm{J}(a, b, y)$.

(27) If $b \not\perp a$ and $l \neq 0_F$, then $\mathrm{J}(a, l \cdot b, x) = l^{-1} \cdot \mathrm{J}(a, b, x)$.

(28) If $b \not\perp a$ and $l \neq 0_F$, then $\mathrm{J}(l \cdot a, b, x) = \mathrm{J}(a, b, x)$.

(29) If $b \not\perp a$ and $p \perp a$, then $\mathrm{J}(a, b + p, c) = \mathrm{J}(a, b, c)$ and $\mathrm{J}(a, b, c + p) = \mathrm{J}(a, b, c)$.

(30) If $b \not\perp a$ and $p \perp b$ and $p \perp c$, then $\mathrm{J}(a + p, b, c) = \mathrm{J}(a, b, c)$.

(31) If $b \not\perp a$ and $c - b \perp a$, then $\mathrm{J}(a, b, c) = \mathbf{1}_F$.

(32) If $b \not\perp a$, then $\mathrm{J}(a, b, b) = \mathbf{1}_F$.

(33) If $b \not\perp a$, then $x \perp a$ iff $\mathrm{J}(a, b, x) = 0_F$.

(34) If $b \not\perp a$ and $q \not\perp a$, then $\mathrm{J}(a, b, p) \cdot \mathrm{J}(a, b, q)^{-1} = \mathrm{J}(a, q, p)$.

(35) If $b \not\perp a$ and $c \not\perp a$, then $\mathrm{J}(a, b, c) = \mathrm{J}(a, c, b)^{-1}$.

(36) If $b \not\perp a$ and $b \perp c + a$, then $\mathrm{J}(a, b, c) = -\mathrm{J}(c, b, a)$.

(37) If $a \not\perp b$ and $c \not\perp b$, then $\mathrm{J}(c, b, a) = \mathrm{J}(b, a, c)^{-1} \cdot \mathrm{J}(a, b, c)$.

(38) If $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$, then $\mathrm{J}(a, q, p) \cdot \mathrm{J}(p, a, x) = \mathrm{J}(q, a, x) \cdot \mathrm{J}(x, q, p)$.

---

[2] The propositions (1)–(10) have been removed.
[3] The propositions (17) and (18) have been removed.
[4] The definitions (Def. 3)–(Def. 5) have been removed.
[5] The proposition (23) has been removed.

(39) If $p \not\perp a$ and $p \not\perp x$ and $q \not\perp a$ and $q \not\perp x$ and $b \not\perp a$, then $\mathrm{J}(a,b,p) \cdot \mathrm{J}(p,a,x) \cdot \mathrm{J}(x,p,y) = \mathrm{J}(a,b,q) \cdot \mathrm{J}(q,a,x) \cdot \mathrm{J}(x,q,y)$.

(40) If $a \not\perp p$ and $x \not\perp p$ and $y \not\perp p$, then $\mathrm{J}(p,a,x) \cdot \mathrm{J}(x,p,y) = \mathrm{J}(p,a,y) \cdot \mathrm{J}(y,p,x)$.

Let us consider F, S, x, y, a, b. Let us assume that $b \not\perp a$. The functor $x \cdot_{a,b} y$ yields an element of F and is defined as follows:

(Def. 7)(i) For every q such that $q \not\perp a$ and $q \not\perp x$ holds $x \cdot_{a,b} y = \mathrm{J}(a,b,q) \cdot \mathrm{J}(q,a,x) \cdot \mathrm{J}(x,q,y)$ if there exists p such that $p \not\perp a$ and $p \not\perp x$,

(ii) $x \cdot_{a,b} y = 0_F$ if for every p holds $p \perp a$ or $p \perp x$.

We now state several propositions:

(43)[6] If $b \not\perp a$ and $x = 0_S$, then $x \cdot_{a,b} y = 0_F$.

(44) If $b \not\perp a$, then $x \cdot_{a,b} y = 0_F$ iff $y \perp x$.

(45) If $b \not\perp a$, then $x \cdot_{a,b} y = y \cdot_{a,b} x$.

(46) If $b \not\perp a$, then $x \cdot_{a,b} (l \cdot y) = l \cdot x \cdot_{a,b} y$.

(47) If $b \not\perp a$, then $x \cdot_{a,b} (y+z) = x \cdot_{a,b} y + x \cdot_{a,b} z$.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Construction of a bilinear antisymmetric form in symplectic vector space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/symsp_1.html.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[7] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[8] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[9] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received November 23, 1989

Published January 2, 2004

---

[6] The propositions (41) and (42) have been removed.