

# Many Sorted Quotient Algebra

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**Summary.** This article introduces the construction of a many sorted quotient algebra. A few preliminary notions such as a many sorted relation, a many sorted equivalence relation, a many sorted congruence and the set of all classes of a many sorted relation are also formulated.

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The articles [10], [14], [15], [3], [16], [5], [9], [4], [2], [1], [13], [11], [7], [12], [6], and [8] provide the notation and terminology for this paper.

## 1. MANY SORTED RELATION

In this paper  $S$  denotes a non void non empty many sorted signature,  $U_1$  denotes an algebra over  $S$ ,  $o$  denotes an operation symbol of  $S$ , and  $s$  denotes a sort symbol of  $S$ .

Let  $I_1$  be a function. We say that  $I_1$  is binary relation yielding if and only if:

(Def. 1) For every set  $x$  such that  $x \in \text{dom } I_1$  holds  $I_1(x)$  is a binary relation.

Let  $I$  be a set. Observe that there exists a many sorted set indexed by  $I$  which is binary relation yielding.

Let  $I$  be a set. A many sorted relation indexed by  $I$  is a binary relation yielding many sorted set indexed by  $I$ .

Let  $I$  be a set and let  $A, B$  be many sorted sets indexed by  $I$ . A many sorted set indexed by  $I$  is said to be a many sorted relation between  $A$  and  $B$  if:

(Def. 2) For every set  $i$  such that  $i \in I$  holds  $it(i)$  is a relation between  $A(i)$  and  $B(i)$ .

Let  $I$  be a set and let  $A, B$  be many sorted sets indexed by  $I$ . Observe that every many sorted relation between  $A$  and  $B$  is binary relation yielding.

Let  $I$  be a set and let  $A$  be a many sorted set indexed by  $I$ . A many sorted relation indexed by  $A$  is a many sorted relation between  $A$  and  $A$ .

Let  $I$  be a set, let  $A$  be a many sorted set indexed by  $I$ , and let  $I_1$  be a many sorted relation indexed by  $A$ . We say that  $I_1$  is equivalence if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let  $i$  be a set and  $R$  be a binary relation on  $A(i)$ . If  $i \in I$  and  $I_1(i) = R$ , then  $R$  is an equivalence relation of  $A(i)$ .

Let  $I$  be a non empty set, let  $A, B$  be many sorted sets indexed by  $I$ , let  $F$  be a many sorted relation between  $A$  and  $B$ , and let  $i$  be an element of  $I$ . Then  $F(i)$  is a relation between  $A(i)$  and  $B(i)$ .

Let  $S$  be a non empty many sorted signature and let  $U_1$  be an algebra over  $S$ . A many sorted relation indexed by  $U_1$  is a many sorted relation indexed by the sorts of  $U_1$ .

Let  $S$  be a non empty many sorted signature, let  $U_1$  be an algebra over  $S$ , and let  $I_1$  be a many sorted relation indexed by  $U_1$ . We say that  $I_1$  is equivalence if and only if:

(Def. 5)<sup>1</sup>  $I_1$  is equivalence.

Let  $S$  be a non void non empty many sorted signature and let  $U_1$  be an algebra over  $S$ . One can check that there exists a many sorted relation indexed by  $U_1$  which is equivalence.

The following proposition is true

(1) Let  $R$  be an equivalence many sorted relation indexed by  $U_1$ . Then  $R(s)$  is an equivalence relation of (the sorts of  $U_1$ )( $s$ ).

Let  $S$  be a non void non empty many sorted signature, let  $U_1$  be a non-empty algebra over  $S$ , and let  $R$  be an equivalence many sorted relation indexed by  $U_1$ . We say that  $R$  is MSCongruence-like if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let  $o$  be an operation symbol of  $S$  and  $x, y$  be elements of  $\text{Args}(o, U_1)$ . Suppose that for every natural number  $n$  such that  $n \in \text{dom } x$  holds  $\langle x(n), y(n) \rangle \in R(\text{Arity}(o)_n)$ . Then  $\langle (\text{Den}(o, U_1))(x), (\text{Den}(o, U_1))(y) \rangle \in R(\text{the result sort of } o)$ .

Let  $S$  be a non void non empty many sorted signature and let  $U_1$  be a non-empty algebra over  $S$ . One can check that there exists an equivalence many sorted relation indexed by  $U_1$  which is MSCongruence-like.

Let  $S$  be a non void non empty many sorted signature and let  $U_1$  be a non-empty algebra over  $S$ . A congruence of  $U_1$  is a MSCongruence-like equivalence many sorted relation indexed by  $U_1$ .

Let  $S$  be a non void non empty many sorted signature, let  $U_1$  be an algebra over  $S$ , let  $R$  be an equivalence many sorted relation indexed by  $U_1$ , and let  $i$  be an element of  $S$ . Then  $R(i)$  is an equivalence relation of (the sorts of  $U_1$ )( $i$ ).

Let  $S$  be a non void non empty many sorted signature, let  $U_1$  be an algebra over  $S$ , let  $R$  be an equivalence many sorted relation indexed by  $U_1$ , let  $i$  be an element of  $S$ , and let  $x$  be an element of (the sorts of  $U_1$ )( $i$ ). The functor  $[x]_R$  yielding a subset of (the sorts of  $U_1$ )( $i$ ) is defined as follows:

(Def. 7)  $[x]_R = [x]_{R(i)}$ .

Let us consider  $S$ , let  $U_1$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $U_1$ . The functor  $\text{Classes } R$  yielding a non-empty many sorted set indexed by the carrier of  $S$  is defined by:

(Def. 8) For every element  $s$  of  $S$  holds  $(\text{Classes } R)(s) = \text{Classes } R(s)$ .

## 2. MANY SORTED QUOTIENT ALGEBRA

Let us consider  $S$ , let  $M_1, M_2$  be many sorted sets indexed by the operation symbols of  $S$ , let  $F$  be a many sorted function from  $M_1$  into  $M_2$ , and let  $o$  be an operation symbol of  $S$ . Then  $F(o)$  is a function from  $M_1(o)$  into  $M_2(o)$ .

Let  $I$  be a non empty set, let  $p$  be a finite sequence of elements of  $I$ , and let  $X$  be a non-empty many sorted set indexed by  $I$ . Then  $X \cdot p$  is a non-empty many sorted set indexed by  $\text{dom } p$ .

Let us consider  $S, o$ , let  $A$  be a non-empty algebra over  $S$ , let  $R$  be a congruence of  $A$ , and let  $x$  be an element of  $\text{Args}(o, A)$ . The functor  $R\#x$  yields an element of  $\prod(\text{Classes } R \cdot \text{Arity}(o))$  and is defined by:

(Def. 9) For every natural number  $n$  such that  $n \in \text{dom } \text{Arity}(o)$  holds  $(R\#x)(n) = [x(n)]_{R(\text{Arity}(o)_n)}$ .

Let us consider  $S, o$ , let  $A$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $A$ . The functor  $\text{QuotRes}(R, o)$  yielding a function from  $((\text{the sorts of } A) \cdot (\text{the result sort of } S))(o)$  into  $(\text{Classes } R \cdot \text{the result sort of } S)(o)$  is defined by:

(Def. 10) For every element  $x$  of  $(\text{the sorts of } A)(\text{the result sort of } o)$  holds  $(\text{QuotRes}(R, o))(x) = [x]_R$ .

The functor  $\text{QuotArgs}(R, o)$  yields a function from  $((\text{the sorts of } A)^\# \cdot \text{the arity of } S)(o)$  into  $((\text{Classes } R)^\# \cdot \text{the arity of } S)(o)$  and is defined as follows:

(Def. 11) For every element  $x$  of  $\text{Args}(o, A)$  holds  $(\text{QuotArgs}(R, o))(x) = R\#x$ .

<sup>1</sup> The definition (Def. 4) has been removed.

Let us consider  $S$ , let  $A$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $A$ . The functor  $\text{QuotRes}(R)$  yielding a many sorted function from (the sorts of  $A$ )  $\cdot$  (the result sort of  $S$ ) into  $\text{Classes } R \cdot$  the result sort of  $S$  is defined as follows:

(Def. 12) For every operation symbol  $o$  of  $S$  holds  $(\text{QuotRes}(R))(o) = \text{QuotRes}(R, o)$ .

The functor  $\text{QuotArgs}(R)$  yields a many sorted function from (the sorts of  $A$ )<sup>#</sup>  $\cdot$  the arity of  $S$  into  $(\text{Classes } R)$ <sup>#</sup>  $\cdot$  the arity of  $S$  and is defined by:

(Def. 13) For every operation symbol  $o$  of  $S$  holds  $(\text{QuotArgs}(R))(o) = \text{QuotArgs}(R, o)$ .

Next we state the proposition

(2) Let  $A$  be a non-empty algebra over  $S$ ,  $R$  be a congruence of  $A$ , and  $x$  be a set. Suppose  $x \in ((\text{Classes } R)$ <sup>#</sup>  $\cdot$  the arity of  $S$ )( $o$ ). Then there exists an element  $a$  of  $\text{Args}(o, A)$  such that  $x = R\#a$ .

Let us consider  $S$ ,  $o$ , let  $A$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $A$ . The functor  $\text{QuotCharact}(R, o)$  yields a function from  $((\text{Classes } R)$ <sup>#</sup>  $\cdot$  the arity of  $S$ )( $o$ ) into  $(\text{Classes } R \cdot$  the result sort of  $S$ )( $o$ ) and is defined by:

(Def. 14) For every element  $a$  of  $\text{Args}(o, A)$  such that  $R\#a \in ((\text{Classes } R)$ <sup>#</sup>  $\cdot$  the arity of  $S$ )( $o$ ) holds  $(\text{QuotCharact}(R, o))(R\#a) = (\text{QuotRes}(R, o) \cdot \text{Den}(o, A))(a)$ .

Let us consider  $S$ , let  $A$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $A$ . The functor  $\text{QuotCharact}(R)$  yields a many sorted function from  $(\text{Classes } R)$ <sup>#</sup>  $\cdot$  the arity of  $S$  into  $\text{Classes } R \cdot$  the result sort of  $S$  and is defined as follows:

(Def. 15) For every operation symbol  $o$  of  $S$  holds  $(\text{QuotCharact}(R))(o) = \text{QuotCharact}(R, o)$ .

Let us consider  $S$ , let  $U_1$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $U_1$ . The functor  $U_1/R$  yields an algebra over  $S$  and is defined as follows:

(Def. 16)  $U_1/R = \langle \text{Classes } R, \text{QuotCharact}(R) \rangle$ .

Let us consider  $S$ , let  $U_1$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $U_1$ . One can verify that  $U_1/R$  is strict and non-empty.

Let us consider  $S$ , let  $U_1$  be a non-empty algebra over  $S$ , let  $R$  be a congruence of  $U_1$ , and let  $s$  be a sort symbol of  $S$ . The functor  $\text{MSNatHom}(U_1, R, s)$  yielding a function from (the sorts of  $U_1$ )( $s$ ) into  $(\text{Classes } R)$ ( $s$ ) is defined by:

(Def. 17) For every set  $x$  such that  $x \in (\text{the sorts of } U_1)(s)$  holds  $(\text{MSNatHom}(U_1, R, s))(x) = [x]_{R(s)}$ .

Let us consider  $S$ , let  $U_1$  be a non-empty algebra over  $S$ , and let  $R$  be a congruence of  $U_1$ . The functor  $\text{MSNatHom}(U_1, R)$  yielding a many sorted function from  $U_1$  into  $U_1/R$  is defined as follows:

(Def. 18) For every sort symbol  $s$  of  $S$  holds  $(\text{MSNatHom}(U_1, R))(s) = \text{MSNatHom}(U_1, R, s)$ .

One can prove the following proposition

(3) For every non-empty algebra  $U_1$  over  $S$  and for every congruence  $R$  of  $U_1$  holds  $\text{MSNatHom}(U_1, R)$  is an epimorphism of  $U_1$  onto  $U_1/R$ .

Let us consider  $S$ , let  $U_1, U_2$  be non-empty algebras over  $S$ , let  $F$  be a many sorted function from  $U_1$  into  $U_2$ , and let  $s$  be a sort symbol of  $S$ . The functor  $\text{Congruence}(F, s)$  yielding an equivalence relation of (the sorts of  $U_1$ )( $s$ ) is defined by:

(Def. 19) For all elements  $x, y$  of (the sorts of  $U_1$ )( $s$ ) holds  $\langle x, y \rangle \in \text{Congruence}(F, s)$  iff  $F(s)(x) = F(s)(y)$ .

Let us consider  $S$ , let  $U_1, U_2$  be non-empty algebras over  $S$ , and let  $F$  be a many sorted function from  $U_1$  into  $U_2$ . Let us assume that  $F$  is a homomorphism of  $U_1$  into  $U_2$ . The functor  $\text{Congruence}(F)$  yields a congruence of  $U_1$  and is defined as follows:

(Def. 20) For every sort symbol  $s$  of  $S$  holds  $(\text{Congruence}(F))(s) = \text{Congruence}(F, s)$ .

Let us consider  $S$ , let  $U_1, U_2$  be non-empty algebras over  $S$ , let  $F$  be a many sorted function from  $U_1$  into  $U_2$ , and let  $s$  be a sort symbol of  $S$ . Let us assume that  $F$  is a homomorphism of  $U_1$  into  $U_2$ . The functor  $\text{MSHomQuot}(F, s)$  yields a function from (the sorts of  $U_1/\text{Congruence}(F)$ )( $s$ ) into (the sorts of  $U_2$ )( $s$ ) and is defined as follows:

(Def. 21) For every element  $x$  of (the sorts of  $U_1$ )( $s$ ) holds  $(\text{MSHomQuot}(F, s))(x)_{\text{Congruence}(F, s)} = F(s)(x)$ .

Let us consider  $S$ , let  $U_1, U_2$  be non-empty algebras over  $S$ , and let  $F$  be a many sorted function from  $U_1$  into  $U_2$ . The functor  $\text{MSHomQuot}(F)$  yields a many sorted function from  $U_1/\text{Congruence}(F)$  into  $U_2$  and is defined by:

(Def. 22) For every sort symbol  $s$  of  $S$  holds  $(\text{MSHomQuot}(F))(s) = \text{MSHomQuot}(F, s)$ .

We now state three propositions:

- (4) Let  $U_1, U_2$  be non-empty algebras over  $S$  and  $F$  be a many sorted function from  $U_1$  into  $U_2$ . Suppose  $F$  is a homomorphism of  $U_1$  into  $U_2$ . Then  $\text{MSHomQuot}(F)$  is a monomorphism of  $U_1/\text{Congruence}(F)$  into  $U_2$ .
- (5) Let  $U_1, U_2$  be non-empty algebras over  $S$  and  $F$  be a many sorted function from  $U_1$  into  $U_2$ . Suppose  $F$  is an epimorphism of  $U_1$  onto  $U_2$ . Then  $\text{MSHomQuot}(F)$  is an isomorphism of  $U_1/\text{Congruence}(F)$  and  $U_2$ .
- (6) Let  $U_1, U_2$  be non-empty algebras over  $S$  and  $F$  be a many sorted function from  $U_1$  into  $U_2$ . If  $F$  is an epimorphism of  $U_1$  onto  $U_2$ , then  $U_1/\text{Congruence}(F)$  and  $U_2$  are isomorphic.

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