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Free Many Sorted Universal Algebra

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The articles [17], [11], [22], [23], [24], [9], [18], [10], [6], [12], [3], [16], [1], [21], [13], [4], [2], [5], [7], [19], [15], [20], [8], and [14] provide the notation and terminology for this paper.

1. Preliminaries

The following proposition is true

(1) Let I be a set, J be a non empty set, f be a function from I into J^*, X be a many sorted set indexed by J, p be an element of J^*, and x be a set. If $x \in I$ and $p = f(x)$, then $(X^\# \cdot f)(x) = \prod(X \cdot p)$.

Let I be a set, let A, B be many sorted sets indexed by I, let C be a many sorted subset indexed by A, and let F be a many sorted function from A into B. The functor $F \restriction C$ yielding a many sorted function from C into B is defined by:

(Def. 1) For every set i such that $i \in I$ and for every function f from $A(i)$ into $B(i)$ such that $f = F(i)$ holds $(F \restriction C)(i) = f \restriction C(i)$.

Let I be a set, let X be a many sorted set indexed by I, and let i be a set. Let us assume that $i \in I$. The functor $\mathrm{coprod}(i,X)$ yielding a set is defined by:

(Def. 2) For every set x holds $x \in \mathrm{coprod}(i,X)$ iff there exists a set a such that $a \in X(i)$ and $x = \langle a, i\rangle$.

Let I be a set and let X be a many sorted set indexed by I. Then $\mathrm{disjoint}X$ is a many sorted set indexed by I and it can be characterized by the condition:

(Def. 3) For every set i such that $i \in I$ holds $(\mathrm{disjoint}X)(i) = \mathrm{coprod}(i,X)$.

We introduce $\mathrm{coprod}(X)$ as a synonym of $\mathrm{disjoint}X$.

Let I be a non empty set and let X be a non-empty many sorted set indexed by I. Note that $\mathrm{coprod}(X)$ is non-empty.

Let I be a non empty set and let X be a non-empty many sorted set indexed by I. Observe that $\bigcup X$ is non empty.

The following proposition is true

(2) Let I be a set, X be a many sorted set indexed by I, and i be a set. If $i \in I$, then $X(i) \neq \emptyset$ iff $(\mathrm{coprod}(X))(i) \neq \emptyset$.

2. Free Many Sorted Universal Algebra — General Notions

In the sequel S denotes a non void non empty many sorted signature and U_0 denotes an algebra over S.

Let S be a non void non empty many sorted signature and let U_0 be an algebra over S. A subset of U_0 is called a generator set of U_0 if:

(Def. 4) The sorts of $\mathrm{Gen}(\mathrm{it}) =$ the sorts of U_0.

The following proposition is true

(3) Let S be a non void non empty many sorted signature, U_0 be a strict non-empty algebra over S, and A be a subset of U_0. Then A is a generator set of U_0 if and only if $\mathrm{Gen}(A) = U_0$.

Let us consider S, U_0 and let I_1 be a generator set of U_0. We say that I_1 is free if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let U_1 be a non-empty algebra over S and f be a many sorted function from I_1 into the sorts of U_1. Then there exists a many sorted function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \restriction I_1 = f$.

Let S be a non void non empty many sorted signature and let I_1 be an algebra over S. We say that I_1 is free if and only if:

(Def. 6) There exists a generator set of I_1 which is free.

We now state the proposition

(4) Let S be a non void non empty many sorted signature and X be a many sorted set indexed by the carrier of S. Then $\bigcup \mathrm{coprod}(X)$ misses $[\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$.

3. Construction of Free Many Sorted Algebra

Let S be a non void many sorted signature. Note that the operation symbols of S is non empty.

Let S be a non void non empty many sorted signature and let X be a many sorted set indexed by the carrier of S. The functor $\mathrm{REL}(X)$ yielding a relation between $[\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X)$ and $([\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^*$ is defined by the condition (Def. 9).

(Def. 9)[1] Let a be an element of $[\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X)$ and b be an element of $([\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^*$. Then $\langle a, b\rangle \in \mathrm{REL}(X)$ if and only if the following conditions are satisfied:

(i) $a \in [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$, and

(ii) for every operation symbol o of S such that $\langle o,$ the carrier of $S\rangle = a$ holds $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$ and for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [\!:$the operation symbols of S, $\{$the carrier of $S\}:\!]$, then for every operation symbol o_1 of S such that $\langle o_1,$ the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

In the sequel S denotes a non void non empty many sorted signature, X denotes a many sorted set indexed by the carrier of S, o denotes an operation symbol of S, and b denotes an element of $([\!:$the operation symbols of S, $\{$the carrier of $S\}:\!] \cup \bigcup \mathrm{coprod}(X))^*$.

One can prove the following proposition

[1] The definitions (Def. 7) and (Def. 8) have been removed.

(5) $\langle\langle o$, the carrier of $S\rangle, b\rangle \in \mathrm{REL}(X)$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, b = \mathrm{len}\, \mathrm{Arity}(o)$, and

(ii) for every set x such that $x \in \mathrm{dom}\, b$ holds if $b(x) \in [$: the operation symbols of S, {the carrier of S} :], then for every operation symbol o_1 of S such that $\langle o_1$, the carrier of $S\rangle = b(x)$ holds the result sort of $o_1 = \mathrm{Arity}(o)(x)$ and if $b(x) \in \bigcup \mathrm{coprod}(X)$, then $b(x) \in \mathrm{coprod}(\mathrm{Arity}(o)(x), X)$.

Let S be a non void non empty many sorted signature and let X be a many sorted set indexed by the carrier of S. The functor $\mathrm{DTConMSA}(X)$ yields a tree construction structure and is defined by:

(Def. 10) $\mathrm{DTConMSA}(X) = \langle [$: the operation symbols of S, {the carrier of S} :] $\cup \bigcup \mathrm{coprod}(X)$, $\mathrm{REL}(X)\rangle$.

Let S be a non void non empty many sorted signature and let X be a many sorted set indexed by the carrier of S. Observe that $\mathrm{DTConMSA}(X)$ is strict and non empty.

The following proposition is true

(6) Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then the nonterminals of $\mathrm{DTConMSA}(X) = [$: the operation symbols of S, {the carrier of S} :] and the terminals of $\mathrm{DTConMSA}(X) = \bigcup \mathrm{coprod}(X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Observe that $\mathrm{DTConMSA}(X)$ has terminals, nonterminals, and useful nonterminals.

One can prove the following proposition

(7) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, and t be a set. Then $t \in$ the terminals of $\mathrm{DTConMSA}(X)$ if and only if there exists a sort symbol s of S and there exists a set x such that $x \in X(s)$ and $t = \langle x, s\rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S. The functor $\mathrm{Sym}(o, X)$ yields a symbol of $\mathrm{DTConMSA}(X)$ and is defined by:

(Def. 11) $\mathrm{Sym}(o, X) = \langle o$, the carrier of $S\rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let s be a sort symbol of S. The functor $\mathrm{FreeSort}(X, s)$ yielding a subset of $\mathrm{TS}(\mathrm{DTConMSA}(X))$ is defined by the condition (Def. 12).

(Def. 12) $\mathrm{FreeSort}(X, s) = \{a; a$ ranges over elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$: $\bigvee_{x:\mathrm{set}} (x \in X(s) \wedge a =$ the root tree of $\langle x, s\rangle) \vee \bigvee_{o:\text{operation symbol of } S} (\langle o$, the carrier of $S\rangle = a(\emptyset) \wedge$ the result sort of $o = s)\}$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let s be a sort symbol of S. Note that $\mathrm{FreeSort}(X, s)$ is non empty.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. The functor $\mathrm{FreeSorts}(X)$ yields a many sorted set indexed by the carrier of S and is defined as follows:

(Def. 13) For every sort symbol s of S holds $(\mathrm{FreeSorts}(X))(s) = \mathrm{FreeSort}(X, s)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that $\mathrm{FreeSorts}(X)$ is non-empty.

One can prove the following propositions:

(8) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, o be an operation symbol of S, and x be a set. Suppose $x \in ((\mathrm{FreeSorts}(X))^{\#} \cdot$ the arity of $S)(o)$. Then x is a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$.

(9) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, o be an operation symbol of S, and p be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$. Then $p \in ((\mathrm{FreeSorts}(X))^{\#} \cdot \text{the arity of } S)(o)$ if and only if $\mathrm{dom}\, p = \mathrm{dom}\,\mathrm{Arity}(o)$ and for every natural number n such that $n \in \mathrm{dom}\, p$ holds $p(n) \in \mathrm{FreeSort}(X, \mathrm{Arity}(o)_n)$.

(10) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, o be an operation symbol of S, and p be a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$. Then $\mathrm{Sym}(o,X) \Rightarrow$ the roots of p if and only if $p \in ((\mathrm{FreeSorts}(X))^{\#} \cdot \text{the arity of } S)(o)$.

(12)[2] Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\bigcup \mathrm{rng}\,\mathrm{FreeSorts}(X) = \mathrm{TS}(\mathrm{DTConMSA}(X))$.

(13) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, and s_1, s_2 be sort symbols of S. If $s_1 \neq s_2$, then $(\mathrm{FreeSorts}(X))(s_1)$ misses $(\mathrm{FreeSorts}(X))(s_2)$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let o be an operation symbol of S. The functor $\mathrm{DenOp}(o,X)$ yielding a function from $((\mathrm{FreeSorts}(X))^{\#} \cdot \text{the arity of } S)(o)$ into $(\mathrm{FreeSorts}(X) \cdot \text{the result sort of } S)(o)$ is defined as follows:

(Def. 14) For every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConMSA}(X))$ such that $\mathrm{Sym}(o,X) \Rightarrow$ the roots of p holds $(\mathrm{DenOp}(o,X))(p) = \mathrm{Sym}(o,X)\text{-tree}(p)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. The functor $\mathrm{FreeOperations}(X)$ yields a many sorted function from $(\mathrm{FreeSorts}(X))^{\#} \cdot$ the arity of S into $\mathrm{FreeSorts}(X) \cdot$ the result sort of S and is defined as follows:

(Def. 15) For every operation symbol o of S holds $(\mathrm{FreeOperations}(X))(o) = \mathrm{DenOp}(o,X)$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. The functor $\mathrm{Free}(X)$ yielding an algebra over S is defined by:

(Def. 16) $\mathrm{Free}(X) = \langle \mathrm{FreeSorts}(X), \mathrm{FreeOperations}(X) \rangle$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. Note that $\mathrm{Free}(X)$ is strict and non-empty.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let s be a sort symbol of S. The functor $\mathrm{FreeGenerator}(s,X)$ yields a subset of $(\mathrm{FreeSorts}(X))(s)$ and is defined as follows:

(Def. 17) For every set x holds $x \in \mathrm{FreeGenerator}(s,X)$ iff there exists a set a such that $a \in X(s)$ and $x =$ the root tree of $\langle a, s \rangle$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let s be a sort symbol of S. Observe that $\mathrm{FreeGenerator}(s,X)$ is non empty.

We now state the proposition

(14) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, and s be a sort symbol of S. Then $\mathrm{FreeGenerator}(s,X) = \{$the root tree of t; t ranges over symbols of $\mathrm{DTConMSA}(X)$: $t \in$ the terminals of $\mathrm{DTConMSA}(X) \wedge t_{\mathbf{2}} = s\}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. The functor $\mathrm{FreeGenerator}(X)$ yielding a generator set of $\mathrm{Free}(X)$ is defined by:

[2] The proposition (11) has been removed.

(Def. 18) For every sort symbol s of S holds $(\mathrm{FreeGenerator}(X))(s) = \mathrm{FreeGenerator}(s,X)$.

The following two propositions are true:

(15) Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\mathrm{FreeGenerator}(X)$ is non-empty.

(16) Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\bigcup \mathrm{rng}\,\mathrm{FreeGenerator}(X) = \{$the root tree of t; t ranges over symbols of $\mathrm{DTConMSA}(X)$: $t \in$ the terminals of $\mathrm{DTConMSA}(X)\}$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let s be a sort symbol of S. The functor $\mathrm{Reverse}(s,X)$ yielding a function from $\mathrm{FreeGenerator}(s,X)$ into $X(s)$ is defined by:

(Def. 19) For every symbol t of $\mathrm{DTConMSA}(X)$ such that the root tree of $t \in \mathrm{FreeGenerator}(s,X)$ holds $(\mathrm{Reverse}(s,X))(\text{the root tree of } t) = t_{\mathbf{1}}$.

Let S be a non void non empty many sorted signature and let X be a non-empty many sorted set indexed by the carrier of S. The functor $\mathrm{Reverse}(X)$ yields a many sorted function from $\mathrm{FreeGenerator}(X)$ into X and is defined as follows:

(Def. 20) For every sort symbol s of S holds $(\mathrm{Reverse}(X))(s) = \mathrm{Reverse}(s,X)$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, let A be a non-empty many sorted set indexed by the carrier of S, let F be a many sorted function from $\mathrm{FreeGenerator}(X)$ into A, and let t be a symbol of $\mathrm{DTConMSA}(X)$. Let us assume that $t \in$ the terminals of $\mathrm{DTConMSA}(X)$. The functor $\pi(F,A,t)$ yielding an element of $\bigcup A$ is defined by:

(Def. 21) For every function f such that $f = F(t_{\mathbf{2}})$ holds $\pi(F,A,t) = f(\text{the root tree of } t)$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let t be a symbol of $\mathrm{DTConMSA}(X)$. Let us assume that there exists a finite sequence p such that $t \Rightarrow p$. The functor ${}^{@}(X,t)$ yields an operation symbol of S and is defined as follows:

(Def. 22) $\langle {}^{@}(X,t), \text{the carrier of } S\rangle = t$.

Let S be a non void non empty many sorted signature, let U_0 be a non-empty algebra over S, let o be an operation symbol of S, and let p be a finite sequence. Let us assume that $p \in \mathrm{Args}(o,U_0)$. The functor $\pi(o,U_0,p)$ yielding an element of $\bigcup(\text{the sorts of } U_0)$ is defined by:

(Def. 23) $\pi(o,U_0,p) = (\mathrm{Den}(o,U_0))(p)$.

Next we state two propositions:

(17) Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\mathrm{FreeGenerator}(X)$ is free.

(18) Let S be a non void non empty many sorted signature and X be a non-empty many sorted set indexed by the carrier of S. Then $\mathrm{Free}(X)$ is free.

Let S be a non void non empty many sorted signature. Observe that there exists a non-empty algebra over S which is free and strict.

Let S be a non void non empty many sorted signature and let U_0 be a free algebra over S. One can check that there exists a generator set of U_0 which is free.

One can prove the following two propositions:

(19) Let S be a non void non empty many sorted signature and U_1 be a non-empty algebra over S. Then there exists a strict free non-empty algebra U_0 over S such that there exists a many sorted function from U_0 into U_1 which is an epimorphism of U_0 onto U_1.

(20) Let S be a non void non empty many sorted signature and U_1 be a strict non-empty algebra over S. Then there exists a strict free non-empty algebra U_0 over S and there exists a many sorted function F from U_0 into U_1 such that F is an epimorphism of U_0 onto U_1 and $\mathrm{Im}\,F = U_1$.

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