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Matrices. Abelian Group of Matrices

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Summary. The basic conceptions of matrix algebra are introduced. The matrix is introduced as the finite sequence of sequences with the same length, i.e. as a sequence of lines. There are considered matrices over a field, and the fact that these matrices with addition form an Abelian group is proved.

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The articles [11], [6], [13], [14], [4], [5], [2], [10], [8], [3], [7], [12], [9], and [1] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x is a set, i, j, n, m are natural numbers, D is a non empty set, K is a non empty double loop structure, s is a finite sequence, a, a_1, a_2, b_1, b_2, d are elements of D, p, p_1, p_2 are finite sequences of elements of D, and F is an add-associative right zeroed right complementable Abelian non empty double loop structure.

Let f be a finite sequence. We say that f is tabular if and only if:

(Def. 1) There exists a natural number n such that for every x such that $x \in \operatorname{rng} f$ there exists s such that $s = x$ and $\operatorname{len} s = n$.

Let us observe that there exists a finite sequence which is tabular.

One can prove the following propositions:

(1) $\langle\langle d \rangle\rangle$ is tabular.

(2) $m \mapsto (n \mapsto x)$ is tabular.

(3) For every s holds $\langle s \rangle$ is tabular.

(4) For all finite sequences s_1, s_2 such that $\operatorname{len} s_1 = n$ and $\operatorname{len} s_2 = n$ holds $\langle s_1, s_2 \rangle$ is tabular.

(5) $\emptyset$ is tabular.

(6) $\langle \emptyset, \emptyset \rangle$ is tabular.

(7) $\langle \langle a_1 \rangle, \langle a_2 \rangle \rangle$ is tabular.

(8) $\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle$ is tabular.

Let f be a binary relation. We say that f is empty yielding if and only if:

(Def. 2) For every set s such that $s \in \operatorname{rng} f$ holds $\overline{\overline{s}} = 0$.

Let D be a set. One can verify that there exists a finite sequence of elements of D^* which is tabular.

Let D be a set. A matrix over D is a tabular finite sequence of elements of D^*.

Let D be a non empty set. Observe that there exists a matrix over D which is non empty yielding.

We now state the proposition

(9) s is a matrix over D iff there exists n such that for every x such that $x \in \operatorname{rng} s$ there exists p such that $x = p$ and $\operatorname{len} p = n$.

Let us consider D, m, n. A matrix over D is called a matrix over D of dimension $m \times n$ if:

(Def. 3) $\operatorname{len} \mathrm{it} = m$ and for every p such that $p \in \operatorname{rng} \mathrm{it}$ holds $\operatorname{len} p = n$.

Let us consider D, n. A matrix over D of dimension n is a matrix over D of dimension $n \times n$.

Let K be a non empty 1-sorted structure. A matrix over K is a matrix over the carrier of K. Let us consider n. A matrix over K of dimension n is a matrix over the carrier of K of dimension $n \times n$. Let us consider m. A matrix over K of dimension $n \times m$ is a matrix over the carrier of K of dimension $n \times m$.

We now state a number of propositions:

(10) $m \longmapsto (n \longmapsto a)$ is a matrix over D of dimension $m \times n$.

(11) For every finite sequence p of elements of D holds $\langle p \rangle$ is a matrix over D of dimension $1 \times \operatorname{len} p$.

(12) For all p_1, p_2 such that $\operatorname{len} p_1 = n$ and $\operatorname{len} p_2 = n$ holds $\langle p_1, p_2 \rangle$ is a matrix over D of dimension $2 \times n$.

(13) $\emptyset$ is a matrix over D of dimension $0 \times m$.

(14) $\langle \emptyset \rangle$ is a matrix over D of dimension 1×0.

(15) $\langle \langle a \rangle \rangle$ is a matrix over D of dimension 1.

(16) $\langle \emptyset, \emptyset \rangle$ is a matrix over D of dimension 2×0.

(17) $\langle \langle a_1, a_2 \rangle \rangle$ is a matrix over D of dimension 1×2.

(18) $\langle \langle a_1 \rangle, \langle a_2 \rangle \rangle$ is a matrix over D of dimension 2×1.

(19) $\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle$ is a matrix over D of dimension 2.

In the sequel M, M_1, M_2 denote matrices over D.

Let M be a tabular finite sequence. The functor $\operatorname{width} M$ yields a natural number and is defined by:

(Def. 4)(i) There exists s such that $s \in \operatorname{rng} M$ and $\operatorname{len} s = \operatorname{width} M$ if $\operatorname{len} M > 0$,

(ii) $\operatorname{width} M = 0$, otherwise.

Next we state the proposition

(20) If $\operatorname{len} M > 0$, then for every n holds M is a matrix over D of dimension $\operatorname{len} M \times n$ iff $n = \operatorname{width} M$.

Let M be a tabular finite sequence. The indices of M yielding a set is defined as follows:

(Def. 5) The indices of $M = [: \operatorname{dom} M, \operatorname{Seg} \operatorname{width} M :]$.

Let D be a set, let M be a matrix over D, and let us consider i, j. Let us assume that $\langle i, j \rangle \in$ the indices of M. The functor $M \circ (i, j)$ yields an element of D and is defined as follows:

(Def. 6) There exists a finite sequence p of elements of D such that $p = M(i)$ and $M \circ (i, j) = p(j)$.

The following proposition is true

(21) If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $M_1 \circ (i, j) = M_2 \circ (i, j)$, then $M_1 = M_2$.

In this article we present several logical schemes. The scheme *MatrixLambda* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j if $\langle i, j\rangle \in$ the indices of M, then $M \circ (i, j) = \mathcal{F}(i, j)$

for all values of the parameters.

The scheme *MatrixEx* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

There exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j if $\langle i, j\rangle \in$ the indices of M, then $\mathcal{P}[i, j, M \circ (i, j)]$

provided the parameters meet the following requirements:

- For all i, j such that $\langle i, j\rangle \in [:\operatorname{Seg}\mathcal{B}, \operatorname{Seg}\mathcal{C}:]$ and for all elements x_1, x_2 of $\mathcal{A}$ such that $\mathcal{P}[i, j, x_1]$ and $\mathcal{P}[i, j, x_2]$ holds $x_1 = x_2$, and
- For all i, j such that $\langle i, j\rangle \in [:\operatorname{Seg}\mathcal{B}, \operatorname{Seg}\mathcal{C}:]$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[i, j, x]$.

We now state several propositions:

(23)[1] For every matrix M over D of dimension $0 \times m$ holds $\operatorname{len} M = 0$ and $\operatorname{width} M = 0$ and the indices of $M = \emptyset$.

(24) Suppose $n > 0$. Let M be a matrix over D of dimension $n \times m$. Then $\operatorname{len} M = n$ and $\operatorname{width} M = m$ and the indices of $M = [:\operatorname{Seg} n, \operatorname{Seg} m:]$.

(25) For every matrix M over D of dimension n holds $\operatorname{len} M = n$ and $\operatorname{width} M = n$ and the indices of $M = [:\operatorname{Seg} n, \operatorname{Seg} n:]$.

(26) For every matrix M over D of dimension $n \times m$ holds $\operatorname{len} M = n$ and the indices of $M = [:\operatorname{Seg} n, \operatorname{Seg} \operatorname{width} M:]$.

(27) For all matrices M_1, M_2 over D of dimension $n \times m$ holds the indices of $M_1 =$ the indices of M_2.

(28) Let M_1, M_2 be matrices over D of dimension $n \times m$. Suppose that for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $M_1 \circ (i, j) = M_2 \circ (i, j)$. Then $M_1 = M_2$.

(29) Let M_1 be a matrix over D of dimension n and given i, j. If $\langle i, j\rangle \in$ the indices of M_1, then $\langle j, i\rangle \in$ the indices of M_1.

Let us consider D and let M be a matrix over D. The functor M^{T} yielding a matrix over D is defined by the conditions (Def. 7).

(Def. 7)(i) $\operatorname{len}(M^{\mathrm{T}}) = \operatorname{width} M$,

(ii) for all i, j holds $\langle i, j\rangle \in$ the indices of M^{T} iff $\langle j, i\rangle \in$ the indices of M, and

(iii) for all i, j such that $\langle j, i\rangle \in$ the indices of M holds $M^{\mathrm{T}} \circ (i, j) = M \circ (j, i)$.

Let us consider D, M, i. The functor $\operatorname{Line}(M, i)$ yielding a finite sequence of elements of D is defined as follows:

(Def. 8) $\operatorname{len}\operatorname{Line}(M, i) = \operatorname{width} M$ and for every j such that $j \in \operatorname{Seg}\operatorname{width} M$ holds $\operatorname{Line}(M, i)(j) = M \circ (i, j)$.

The functor $M_{\square, i}$ yields a finite sequence of elements of D and is defined by:

[1] The proposition (22) has been removed.

(Def. 9) $\mathrm{len}(M_{\square,i}) = \mathrm{len}\,M$ and for every j such that $j \in \mathrm{dom}\,M$ holds $M_{\square,i}(j) = M \circ (j,i)$.

Let us consider D, let M be a matrix over D, and let us consider i. Then $\mathrm{Line}(M,i)$ is an element of $D^{\mathrm{width}\,M}$. Then $M_{\square,i}$ is an element of $D^{\mathrm{len}\,M}$.

In the sequel A, B denote matrices over K of dimension n.

Let us consider K, n. The functor $K^{n \times n}$ yields a set and is defined as follows:

(Def. 10) $K^{n \times n} = ((\text{the carrier of } K)^n)^n$.

The functor $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$ yields a matrix over K of dimension n and is defined as follows:

(Def. 11) $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n} = n \mapsto (n \mapsto 0_K)$.

The functor $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ yielding a matrix over K of dimension n is defined by the conditions (Def. 12).

(Def. 12)(i) For every i such that $\langle i, i \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n} \circ (i,i) = \mathbf{1}_K$, and

(ii) for all i, j such that $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}$ and $i \neq j$ holds $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n} \circ (i,j) = 0_K$.

Let us consider A. The functor $-A$ yielding a matrix over K of dimension n is defined by:

(Def. 13) For all i, j such that $\langle i, j \rangle \in$ the indices of A holds $(-A) \circ (i,j) = -(A \circ (i,j))$.

Let us consider B. The functor $A+B$ yields a matrix over K of dimension n and is defined as follows:

(Def. 14) For all i, j such that $\langle i, j \rangle \in$ the indices of A holds $(A+B) \circ (i,j) = (A \circ (i,j)) + (B \circ (i,j))$.

Let us consider K, n. Note that $K^{n \times n}$ is non empty.

We now state two propositions:

(30) If $\langle i, j \rangle \in$ the indices of $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n}$, then $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n \times n} \circ (i,j) = 0_K$.

(31) x is an element of $K^{n \times n}$ iff x is a matrix over K of dimension n.

Let us consider K, n. A matrix over K of dimension n is said to be a diagonal n-dimensional matrix over K if:

(Def. 15) For all i, j such that $\langle i, j\rangle \in$ the indices of it and it $\circ(i, j) \neq 0_K$ holds $i = j$.

In the sequel A, B, C denote matrices over F of dimension n.

One can prove the following propositions:

(32) $A+B=B+A$.

(33) $(A+B)+C=A+(B+C)$.

(34) $A+\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_F^{n\times n}=A$.

(35) $A+-A=\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_F^{n\times n}$.

Let us consider F, n. The functor $F_{\rm G}^{n\times n}$ yields a strict Abelian group and is defined by:

(Def. 16) The carrier of $F_{\rm G}^{n\times n}=F^{n\times n}$ and for all A, B holds (the addition of $F_{\rm G}^{n\times n})(A, B)=A+B$ and the zero of $F_{\rm G}^{n\times n}=\begin{pmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}_F^{n\times n}$.

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