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# Domains of Submodules, Join and Meet of Finite Sequences of Submodules and Quotient Modules

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**Summary.** Notions of domains of submodules, join and meet of finite sequences of submodules and quotient modules. A few basic theorems and schemes related to these notions are proved.

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The articles [13], [5], [18], [3], [4], [2], [1], [12], [19], [11], [14], [6], [7], [17], [16], [15], [10], [8], and [9] provide the notation and terminology for this paper.

## 1. SCHEMES

In this article we present several logical schemes. The scheme *ElementEq* deals with a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> Let $X_1$, $X_2$ be elements of $\mathcal{A}$. Suppose for every set $x$ holds $x \in X_1$ iff $\mathcal{P}[x]$ and for every set $x$ holds $x \in X_2$ iff $\mathcal{P}[x]$. Then $X_1 = X_2$

for all values of the parameters.

The scheme *UnOpEq* deals with a non empty set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> Let $f_1$, $f_2$ be unary operations on $\mathcal{A}$. Suppose for every element $a$ of $\mathcal{A}$ holds $f_1(a) = \mathcal{F}(a)$ and for every element $a$ of $\mathcal{A}$ holds $f_2(a) = \mathcal{F}(a)$. Then $f_1 = f_2$

for all values of the parameters.

The scheme *TriOpEq* deals with a non empty set $\mathcal{A}$ and a ternary functor $\mathcal{F}$ yielding a set, and states that:

> Let $f_1$, $f_2$ be ternary operations on $\mathcal{A}$. Suppose for all elements $a$, $b$, $c$ of $\mathcal{A}$ holds $f_1(a, b, c) = \mathcal{F}(a,b,c)$ and for all elements $a$, $b$, $c$ of $\mathcal{A}$ holds $f_2(a, b, c) = \mathcal{F}(a,b,c)$. Then $f_1 = f_2$

for all values of the parameters.

The scheme *QuaOpEq* deals with a non empty set $\mathcal{A}$ and a 4-ary functor $\mathcal{F}$ yielding a set, and states that:

> Let $f_1$, $f_2$ be quadrary operations on $\mathcal{A}$. Suppose for all elements $a$, $b$, $c$, $d$ of $\mathcal{A}$ holds $f_1(a, b, c, d) = \mathcal{F}(a,b,c,d)$ and for all elements $a$, $b$, $c$, $d$ of $\mathcal{A}$ holds $f_2(a, b, c, d) = \mathcal{F}(a,b,c,d)$. Then $f_1 = f_2$

for all values of the parameters.



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The scheme *Fraenkel1 Ex* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a subset $S$ of $\mathcal{B}$ such that $S = \{\mathcal{F}(x); x$ ranges over elements of $\mathcal{A} : \mathcal{P}[x]\}$

for all values of the parameters.

The scheme *Fr 0* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters satisfy the following condition:

- $\mathcal{B} \in \{a; a$ ranges over elements of $\mathcal{A} : \mathcal{P}[a]\}$.

The scheme *Fr 1* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C} \in \mathcal{A}$ iff $\mathcal{P}[\mathcal{C}]$

provided the following condition is satisfied:

- $\mathcal{A} = \{a; a$ ranges over elements of $\mathcal{B} : \mathcal{P}[a]\}$.

The scheme *Fr 2* deals with a set $\mathcal{A}$, a non empty set $\mathcal{B}$, an element $\mathcal{C}$ of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the parameters satisfy the following conditions:

- $\mathcal{C} \in \mathcal{A}$, and
- $\mathcal{A} = \{a; a$ ranges over elements of $\mathcal{B} : \mathcal{P}[a]\}$.

The scheme *Fr 3* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a non empty set $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{A} \in \mathcal{B}$ iff there exists an element $a$ of $\mathcal{C}$ such that $\mathcal{A} = a$ and $\mathcal{P}[a]$

provided the parameters meet the following requirement:

- $\mathcal{B} = \{a; a$ ranges over elements of $\mathcal{C} : \mathcal{P}[a]\}$.

The scheme *Fr 4* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a set $\mathcal{C}$, an element $\mathcal{D}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{D} \in \mathcal{F}(\mathcal{C})$ iff for every element $b$ of $\mathcal{B}$ such that $b \in \mathcal{C}$ holds $\mathcal{P}[\mathcal{D}, b]$

provided the following requirements are met:

- $\mathcal{F}(\mathcal{C}) = \{a; a$ ranges over elements of $\mathcal{A} : \mathcal{Q}[a, \mathcal{C}]\}$, and
- $\mathcal{Q}[\mathcal{D}, \mathcal{C}]$ iff for every element $b$ of $\mathcal{B}$ such that $b \in \mathcal{C}$ holds $\mathcal{P}[\mathcal{D}, b]$.

## 2. AUXILIARY THEOREMS ON FREE-MODULES

For simplicity, we follow the rules: $x$ is a set, $K$ is a ring, $r$ is a scalar of $K$, $V$ is a left module over $K$, $a$, $b$, $a_1$, $a_2$ are vectors of $V$, $A$, $A_1$, $A_2$ are subsets of $V$, $l$ is a linear combination of $A$, $W$ is a subspace of $V$, and $L_1$ is a finite sequence of elements of $\mathrm{Sub}(V)$.

The following propositions are true:

(1) If $K$ is non trivial and $A$ is linearly independent, then $0_V \notin A$.

(2) If $a \notin A$, then $l(a) = 0_K$.

(3) If $K$ is trivial, then for every $l$ holds the support of $l = \emptyset$ and $\mathrm{Lin}(A)$ is trivial.

(4) If $V$ is non trivial, then for every $A$ such that $A$ is base holds $A \neq \emptyset$.

(5) If $A_1 \cup A_2$ is linearly independent and $A_1$ misses $A_2$, then $\mathrm{Lin}(A_1) \cap \mathrm{Lin}(A_2) = \mathbf{0}_V$.

(6) If $A$ is base and $A = A_1 \cup A_2$ and $A_1$ misses $A_2$, then $V$ is the direct sum of $\mathrm{Lin}(A_1)$ and $\mathrm{Lin}(A_2)$.

## 3. DOMAINS OF SUBMODULES

Let us consider $K$, $V$. A non empty set is called a non empty set of submodules of $V$ if:

(Def. 1) If $x \in$ it, then $x$ is a strict subspace of $V$.

Let us consider $K$, $V$. Then $\mathrm{Sub}(V)$ is a non empty set of submodules of $V$.

Let us consider $K$, $V$ and let $D$ be a non empty set of submodules of $V$. We see that the element of $D$ is a strict subspace of $V$.

Let us consider $K$, $V$ and let $D$ be a non empty set of submodules of $V$. Observe that there exists an element of $D$ which is strict.

Let us consider $K$, $V$. Let us assume that $V$ is non trivial. A strict subspace of $V$ is said to be a line of $V$ if:

(Def. 2) There exists $a$ such that $a \neq 0_V$ and it $= \prod^* a$.

Let us consider $K$, $V$. A non empty set is called a non empty set of lines of $V$ if:

(Def. 3) If $x \in$ it, then $x$ is a line of $V$.

Let us consider $K$, $V$. The functor $\mathrm{lines}(V)$ yielding a non empty set of lines of $V$ is defined as follows:

(Def. 4) $x \in \mathrm{lines}(V)$ iff $x$ is a line of $V$.

Let us consider $K$, $V$ and let $D$ be a non empty set of lines of $V$. We see that the element of $D$ is a line of $V$.

Let us consider $K$, $V$. Let us assume that $V$ is non trivial and $V$ is free. A strict subspace of $V$ is said to be a hiperplane of $V$ if:

(Def. 5) There exists $a$ such that $a \neq 0_V$ and $V$ is the direct sum of $\prod^* a$ and it.

Let us consider $K$, $V$. A non empty set is called a non empty set of hiperplanes of $V$ if:

(Def. 6) If $x \in$ it, then $x$ is a hiperplane of $V$.

Let us consider $K$, $V$. The functor $\mathrm{hiperplanes}(V)$ yielding a non empty set of hiperplanes of $V$ is defined by:

(Def. 7) $x \in \mathrm{hiperplanes}(V)$ iff $x$ is a hiperplane of $V$.

Let us consider $K$, $V$ and let $D$ be a non empty set of hiperplanes of $V$. We see that the element of $D$ is a hiperplane of $V$.

## 4. Join and Meet of Finite Sequences of Submodules

Let us consider $K$, $V$, $L_1$. The functor $\sum L_1$ yields an element of $\mathrm{Sub}(V)$ and is defined by:

(Def. 8) $\sum L_1 = \mathrm{SubJoin} V \circledast L_1$.

The functor $\bigcap L_1$ yielding an element of $\mathrm{Sub}(V)$ is defined as follows:

(Def. 9) $\bigcap L_1 = \mathrm{SubMeet} V \circledast L_1$.

Next we state three propositions:

(12)[1] Let $G$ be a lattice. Then the join operation of $G$ is commutative and associative and the meet operation of $G$ is commutative and associative.

(14)[2] $\mathrm{SubJoin} V$ is commutative and associative and $\mathrm{SubJoin} V$ has a unity and $\mathbf{0}_V = \mathbf{1}_{\mathrm{SubJoin} V}$.

(15) $\mathrm{SubMeet} V$ is commutative and associative and $\mathrm{SubMeet} V$ has a unity and $\Omega_V = \mathbf{1}_{\mathrm{SubMeet} V}$.

## 5. Sum of Subsets of Module

Let us consider $K$, $V$, $A_1$, $A_2$. The functor $A_1 + A_2$ yields a subset of $V$ and is defined as follows:

(Def. 10) $x \in A_1 + A_2$ iff there exist $a_1$, $a_2$ such that $a_1 \in A_1$ and $a_2 \in A_2$ and $x = a_1 + a_2$.

[1] The propositions (7)–(11) have been removed.
[2] The proposition (13) has been removed.

## 6. VECTOR OF SUBSET

Let us consider $K$, $V$, $A$. Let us assume that $A \neq \emptyset$. A vector of $V$ is called a vector of $A$ if:

(Def. 11) It is an element of $A$.

We now state three propositions:

(16) If $A_1 \neq \emptyset$ and $A_1 \subseteq A_2$, then for every $x$ such that $x$ is a vector of $A_1$ holds $x$ is a vector of $A_2$.

(17) $a_2 \in a_1 + W$ iff $a_1 - a_2 \in W$.

(18) $a_1 + W = a_2 + W$ iff $a_1 - a_2 \in W$.

Let us consider $K$, $V$, $W$. The functor $V \looparrowleft W$ yields a set and is defined by:

(Def. 12) $x \in V \looparrowleft W$ iff there exists $a$ such that $x = a + W$.

Let us consider $K$, $V$, $W$. Observe that $V \looparrowleft W$ is non empty.
Let us consider $K$, $V$, $W$, $a$. The functor $a \looparrowleft W$ yielding an element of $V \looparrowleft W$ is defined by:

(Def. 13) $a \looparrowleft W = a + W$.

One can prove the following two propositions:

(19) For every element $x$ of $V \looparrowleft W$ there exists $a$ such that $x = a \looparrowleft W$.

(20) $a_1 \looparrowleft W = a_2 \looparrowleft W$ iff $a_1 - a_2 \in W$.

In the sequel $S_1$, $S_2$ denote elements of $V \looparrowleft W$.
Let us consider $K$, $V$, $W$, $S_1$. The functor $-S_1$ yielding an element of $V \looparrowleft W$ is defined as follows:

(Def. 14) If $S_1 = a \looparrowleft W$, then $-S_1 = (-a) \looparrowleft W$.

Let us consider $S_2$. The functor $S_1 + S_2$ yields an element of $V \looparrowleft W$ and is defined by:

(Def. 15) If $S_1 = a_1 \looparrowleft W$ and $S_2 = a_2 \looparrowleft W$, then $S_1 + S_2 = (a_1 + a_2) \looparrowleft W$.

Let us consider $K$, $V$, $W$. The functor $\mathrm{COMPL}(W)$ yields a unary operation on $V \looparrowleft W$ and is defined by:

(Def. 16) $(\mathrm{COMPL}(W))(S_1) = -S_1$.

The functor $\mathrm{ADD}(W)$ yielding a binary operation on $V \looparrowleft W$ is defined by:

(Def. 17) $(\mathrm{ADD}(W))(S_1, S_2) = S_1 + S_2$.

Let us consider $K$, $V$, $W$. The functor $V(W)$ yielding a strict loop structure is defined as follows:

(Def. 18) $V(W) = \langle V \looparrowleft W, \mathrm{ADD}(W), 0_V \looparrowleft W \rangle$.

Let us consider $K$, $V$, $W$. Observe that $V(W)$ is non empty.
The following proposition is true

(21) $a \looparrowleft W$ is an element of $V(W)$.

Let us consider $K$, $V$, $W$, $a$. The functor $a(W)$ yields an element of $V(W)$ and is defined as follows:

(Def. 19) $a(W) = a \looparrowleft W$.

One can prove the following three propositions:

(22) For every element $x$ of $V(W)$ there exists $a$ such that $x = a(W)$.

(23) $a_1(W) = a_2(W)$ iff $a_1 - a_2 \in W$.

(24) $a(W) + b(W) = (a+b)(W)$ and $0_{V(W)} = 0_V(W)$.

Let us consider $K$, $V$, $W$. One can check that $V(W)$ is Abelian, add-associative, right zeroed, and right complementable.

In the sequel $S$ is an element of $V(W)$.

Let us consider $K$, $V$, $W$, $r$, $S$. The functor $r \cdot S$ yields an element of $V(W)$ and is defined as follows:

(Def. 20) If $S = a(W)$, then $r \cdot S = (r \cdot a)(W)$.

Let us consider $K$, $V$, $W$. The functor LMULT$(W)$ yields a function from [: the carrier of $K$, the carrier of $V(W)$ :] into the carrier of $V(W)$ and is defined as follows:

(Def. 21) $(\mathrm{LMULT}(W))(r, S) = r \cdot S$.

## 7. Quotient modules

Let us consider $K$, $V$, $W$. The functor $\frac{V}{W}$ yields a strict vector space structure over $K$ and is defined by:

(Def. 22) $\frac{V}{W} = \langle$the carrier of $V(W)$, the addition of $V(W)$, the zero of $V(W)$, LMULT$(W)\rangle$.

Let us consider $K$, $V$, $W$. One can check that $\frac{V}{W}$ is non empty.

The following propositions are true:

(26)[3] $a(W)$ is a vector of $\frac{V}{W}$.

(27) Every vector of $\frac{V}{W}$ is an element of $V(W)$.

Let us consider $K$, $V$, $W$, $a$. The functor $\frac{a}{W}$ yielding a vector of $\frac{V}{W}$ is defined by:

(Def. 23) $\frac{a}{W} = a(W)$.

One can prove the following propositions:

(28) For every vector $x$ of $\frac{V}{W}$ there exists $a$ such that $x = \frac{a}{W}$.

(29) $\frac{a_1}{W} = \frac{a_2}{W}$ iff $a_1 - a_2 \in W$.

(30) $\frac{a}{W} + \frac{b}{W} = \frac{a+b}{W}$ and $r \cdot \frac{a}{W} = \frac{r \cdot a}{W}$.

(31) $\frac{V}{W}$ is a strict left module over $K$.

Let us consider $K$, $V$, $W$. Observe that $\frac{V}{W}$ is vector space-like.

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