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Curried and Uncurried Functions

Grzegorz Bancerek
Warsaw University
Białystok

Summary. In the article following functors are introduced: the projections of subsets of the Cartesian product, the functor which for every function $f : X \times Y \rightarrow Z$ gives some curried function $(X \rightarrow (Y \rightarrow Z))$, and the functor which from curried functions makes uncurried functions. Some of their properties and some properties of the set of all functions from a set into a set are also shown.

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The articles [9], [6], [11], [12], [3], [2], [4], [10], [1], [5], [8], and [7] provide the notation and terminology for this paper.

We follow the rules: X, Y, Z, X_1, X_2, Y_1, Y_2, x, y, z, t are sets and f, g, f_1, f_2 are functions.

The scheme *LambdaFS* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every g such that $g \in \mathcal{A}$ holds $f(g) = \mathcal{F}(g)$

for all values of the parameters.

We now state the proposition

(1) $\curvearrowleft\emptyset = \emptyset$.

Let us consider X. The functor $\pi_1(X)$ yields a set and is defined by:

(Def. 1) $x \in \pi_1(X)$ iff there exists y such that $\langle x, y \rangle \in X$.

The functor $\pi_2(X)$ yielding a set is defined as follows:

(Def. 2) $y \in \pi_2(X)$ iff there exists x such that $\langle x, y \rangle \in X$.

The following propositions are true:

(4)[1] If $\langle x, y \rangle \in X$, then $x \in \pi_1(X)$ and $y \in \pi_2(X)$.

(5) If $X \subseteq Y$, then $\pi_1(X) \subseteq \pi_1(Y)$ and $\pi_2(X) \subseteq \pi_2(Y)$.

(6) $\pi_1(X \cup Y) = \pi_1(X) \cup \pi_1(Y)$ and $\pi_2(X \cup Y) = \pi_2(X) \cup \pi_2(Y)$.

(7) $\pi_1(X \cap Y) \subseteq \pi_1(X) \cap \pi_1(Y)$ and $\pi_2(X \cap Y) \subseteq \pi_2(X) \cap \pi_2(Y)$.

(8) $\pi_1(X) \setminus \pi_1(Y) \subseteq \pi_1(X \setminus Y)$ and $\pi_2(X) \setminus \pi_2(Y) \subseteq \pi_2(X \setminus Y)$.

(9) $\pi_1(X) \dot{-} \pi_1(Y) \subseteq \pi_1(X \dot{-} Y)$ and $\pi_2(X) \dot{-} \pi_2(Y) \subseteq \pi_2(X \dot{-} Y)$.

[1] The propositions (2) and (3) have been removed.

(10) $\pi_1(\emptyset) = \emptyset$ and $\pi_2(\emptyset) = \emptyset$.

(11) If $Y \neq \emptyset$ or $[: X, Y :] \neq \emptyset$ or $[: Y, X :] \neq \emptyset$, then $\pi_1([: X, Y :]) = X$ and $\pi_2([: Y, X :]) = X$.

(12) $\pi_1([: X, Y :]) \subseteq X$ and $\pi_2([: X, Y :]) \subseteq Y$.

(13) If $Z \subseteq [: X, Y :]$, then $\pi_1(Z) \subseteq X$ and $\pi_2(Z) \subseteq Y$.

(15)[2] $\pi_1(\{\langle x, y \rangle\}) = \{x\}$ and $\pi_2(\{\langle x, y \rangle\}) = \{y\}$.

(16) $\pi_1(\{\langle x, y \rangle, \langle z, t \rangle\}) = \{x, z\}$ and $\pi_2(\{\langle x, y \rangle, \langle z, t \rangle\}) = \{y, t\}$.

(17) If it is not true that there exist x, y such that $\langle x, y \rangle \in X$, then $\pi_1(X) = \emptyset$ and $\pi_2(X) = \emptyset$.

(18) If $\pi_1(X) = \emptyset$ or $\pi_2(X) = \emptyset$, then it is not true that there exist x, y such that $\langle x, y \rangle \in X$.

(19) $\pi_1(X) = \emptyset$ iff $\pi_2(X) = \emptyset$.

(20) $\pi_1(\mathrm{dom}\, f) = \pi_2(\mathrm{dom}{\curvearrowleft} f)$ and $\pi_2(\mathrm{dom}\, f) = \pi_1(\mathrm{dom}{\curvearrowleft} f)$.

(21) For every binary relation f holds $\pi_1(f) = \mathrm{dom}\, f$ and $\pi_2(f) = \mathrm{rng}\, f$.

Let us consider f. The functor curry f yielding a function is defined by the conditions (Def. 3).

(Def. 3)(i) $\mathrm{dom}\,\mathrm{curry}\, f = \pi_1(\mathrm{dom}\, f)$, and

(ii) for every x such that $x \in \pi_1(\mathrm{dom}\, f)$ there exists g such that $(\mathrm{curry}\, f)(x) = g$ and $\mathrm{dom}\, g = \pi_2(\mathrm{dom}\, f \cap [: \{x\}, \pi_2(\mathrm{dom}\, f) :])$ and for every y such that $y \in \mathrm{dom}\, g$ holds $g(y) = f(\langle x, y \rangle)$.

The functor uncurry f yields a function and is defined by the conditions (Def. 4).

(Def. 4)(i) For every t holds $t \in \mathrm{dom}\,\mathrm{uncurry}\, f$ iff there exist x, g, y such that $t = \langle x, y \rangle$ and $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$, and

(ii) for all x, g such that $x \in \mathrm{dom}\,\mathrm{uncurry}\, f$ and $g = f(x_{\mathbf{1}})$ holds $(\mathrm{uncurry}\, f)(x) = g(x_{\mathbf{2}})$.

Let us consider f. The functor curry$'$ f yielding a function is defined as follows:

(Def. 5) $\mathrm{curry}'\, f = \mathrm{curry}{\curvearrowleft} f$.

The functor uncurry$'$ f yielding a function is defined as follows:

(Def. 6) $\mathrm{uncurry}'\, f = {\curvearrowleft}\mathrm{uncurry}\, f$.

One can prove the following propositions:

(26)[3] If $\langle x, y \rangle \in \mathrm{dom}\, f$, then $x \in \mathrm{dom}\,\mathrm{curry}\, f$ and $(\mathrm{curry}\, f)(x)$ is a function.

(27) If $\langle x, y \rangle \in \mathrm{dom}\, f$ and $g = (\mathrm{curry}\, f)(x)$, then $y \in \mathrm{dom}\, g$ and $g(y) = f(\langle x, y \rangle)$.

(28) If $\langle x, y \rangle \in \mathrm{dom}\, f$, then $y \in \mathrm{dom}\,\mathrm{curry}'\, f$ and $(\mathrm{curry}'\, f)(y)$ is a function.

(29) If $\langle x, y \rangle \in \mathrm{dom}\, f$ and $g = (\mathrm{curry}'\, f)(y)$, then $x \in \mathrm{dom}\, g$ and $g(x) = f(\langle x, y \rangle)$.

(30) $\mathrm{dom}\,\mathrm{curry}'\, f = \pi_2(\mathrm{dom}\, f)$.

(31) If $[: X, Y :] \neq \emptyset$ and $\mathrm{dom}\, f = [: X, Y :]$, then $\mathrm{dom}\,\mathrm{curry}\, f = X$ and $\mathrm{dom}\,\mathrm{curry}'\, f = Y$.

(32) If $\mathrm{dom}\, f \subseteq [: X, Y :]$, then $\mathrm{dom}\,\mathrm{curry}\, f \subseteq X$ and $\mathrm{dom}\,\mathrm{curry}'\, f \subseteq Y$.

(33) If $\mathrm{rng}\, f \subseteq Y^X$, then $\mathrm{dom}\,\mathrm{uncurry}\, f = [: \mathrm{dom}\, f, X :]$ and $\mathrm{dom}\,\mathrm{uncurry}'\, f = [: X, \mathrm{dom}\, f :]$.

(34) If it is not true that there exist x, y such that $\langle x, y \rangle \in \mathrm{dom}\, f$, then $\mathrm{curry}\, f = \emptyset$ and $\mathrm{curry}'\, f = \emptyset$.

[2] The proposition (14) has been removed.
[3] The propositions (22)–(25) have been removed.

(35) If it is not true that there exists x such that $x \in \operatorname{dom} f$ and $f(x)$ is a function, then uncurry $f = \emptyset$ and uncurry′ $f = \emptyset$.

(36) Suppose $[:X, Y:] \neq \emptyset$ and $\operatorname{dom} f = [:X, Y:]$ and $x \in X$. Then there exists g such that $(\text{curry} f)(x) = g$ and $\operatorname{dom} g = Y$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and for every y such that $y \in Y$ holds $g(y) = f(\langle x, y\rangle)$.

(37) If $x \in \operatorname{dom}\text{curry} f$, then $(\text{curry} f)(x)$ is a function.

(38) Suppose $x \in \operatorname{dom}\text{curry} f$ and $g = (\text{curry} f)(x)$. Then $\operatorname{dom} g = \pi_2(\operatorname{dom} f \cap [:\{x\}, \pi_2(\operatorname{dom} f):])$ and $\operatorname{dom} g \subseteq \pi_2(\operatorname{dom} f)$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and for every y such that $y \in \operatorname{dom} g$ holds $g(y) = f(\langle x, y\rangle)$ and $\langle x, y\rangle \in \operatorname{dom} f$.

(39) Suppose $[:X, Y:] \neq \emptyset$ and $\operatorname{dom} f = [:X, Y:]$ and $y \in Y$. Then there exists g such that $(\text{curry}' f)(y) = g$ and $\operatorname{dom} g = X$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and for every x such that $x \in X$ holds $g(x) = f(\langle x, y\rangle)$.

(40) If $x \in \operatorname{dom}\text{curry}' f$, then $(\text{curry}' f)(x)$ is a function.

(41) Suppose $x \in \operatorname{dom}\text{curry}' f$ and $g = (\text{curry}' f)(x)$. Then $\operatorname{dom} g = \pi_1(\operatorname{dom} f \cap [:\pi_1(\operatorname{dom} f), \{x\}:])$ and $\operatorname{dom} g \subseteq \pi_1(\operatorname{dom} f)$ and $\operatorname{rng} g \subseteq \operatorname{rng} f$ and for every y such that $y \in \operatorname{dom} g$ holds $g(y) = f(\langle y, x\rangle)$ and $\langle y, x\rangle \in \operatorname{dom} f$.

(42) If $\operatorname{dom} f = [:X, Y:]$, then $\operatorname{rng}\text{curry} f \subseteq (\operatorname{rng} f)^Y$ and $\operatorname{rng}\text{curry}' f \subseteq (\operatorname{rng} f)^X$.

(43) $\operatorname{rng}\text{curry} f \subseteq \pi_2(\operatorname{dom} f) \dot{\to} \operatorname{rng} f$ and $\operatorname{rng}\text{curry}' f \subseteq \pi_1(\operatorname{dom} f) \dot{\to} \operatorname{rng} f$.

(44) If $\operatorname{rng} f \subseteq X \dot{\to} Y$, then $\operatorname{dom}\text{uncurry} f \subseteq [:\operatorname{dom} f, X:]$ and $\operatorname{dom}\text{uncurry}' f \subseteq [:X, \operatorname{dom} f:]$.

(45) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} g$, then $\langle x, y\rangle \in \operatorname{dom}\text{uncurry} f$ and $(\text{uncurry} f)(\langle x, y\rangle) = g(y)$ and $g(y) \in \operatorname{rng}\text{uncurry} f$.

(46) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} g$, then $\langle y, x\rangle \in \operatorname{dom}\text{uncurry}' f$ and $(\text{uncurry}' f)(\langle y, x\rangle) = g(y)$ and $g(y) \in \operatorname{rng}\text{uncurry}' f$.

(47) If $\operatorname{rng} f \subseteq X \dot{\to} Y$, then $\operatorname{rng}\text{uncurry} f \subseteq Y$ and $\operatorname{rng}\text{uncurry}' f \subseteq Y$.

(48) If $\operatorname{rng} f \subseteq Y^X$, then $\operatorname{rng}\text{uncurry} f \subseteq Y$ and $\operatorname{rng}\text{uncurry}' f \subseteq Y$.

(49) $\text{curry}\,\emptyset = \emptyset$ and $\text{curry}'\,\emptyset = \emptyset$.

(50) $\text{uncurry}\,\emptyset = \emptyset$ and $\text{uncurry}'\,\emptyset = \emptyset$.

(51) If $\operatorname{dom} f_1 = [:X, Y:]$ and $\operatorname{dom} f_2 = [:X, Y:]$ and curry $f_1 =$ curry f_2, then $f_1 = f_2$.

(52) If $\operatorname{dom} f_1 = [:X, Y:]$ and $\operatorname{dom} f_2 = [:X, Y:]$ and curry′ $f_1 =$ curry′ f_2, then $f_1 = f_2$.

(53) If $\operatorname{rng} f_1 \subseteq Y^X$ and $\operatorname{rng} f_2 \subseteq Y^X$ and $X \neq \emptyset$ and uncurry $f_1 =$ uncurry f_2, then $f_1 = f_2$.

(54) If $\operatorname{rng} f_1 \subseteq Y^X$ and $\operatorname{rng} f_2 \subseteq Y^X$ and $X \neq \emptyset$ and uncurry′ $f_1 =$ uncurry′ f_2, then $f_1 = f_2$.

(55) If $\operatorname{rng} f \subseteq Y^X$ and $X \neq \emptyset$, then curry uncurry $f = f$ and curry′ uncurry′ $f = f$.

(56) If $\operatorname{dom} f = [:X, Y:]$, then uncurry curry $f = f$ and uncurry′ curry′ $f = f$.

(57) If $\operatorname{dom} f \subseteq [:X, Y:]$, then uncurry curry $f = f$ and uncurry′ curry′ $f = f$.

(58) If $\operatorname{rng} f \subseteq X \dot{\to} Y$ and $\emptyset \notin \operatorname{rng} f$, then curry uncurry $f = f$ and curry′ uncurry′ $f = f$.

(59) If $\operatorname{dom} f_1 \subseteq [:X, Y:]$ and $\operatorname{dom} f_2 \subseteq [:X, Y:]$ and curry $f_1 =$ curry f_2, then $f_1 = f_2$.

(60) If $\operatorname{dom} f_1 \subseteq [:X, Y:]$ and $\operatorname{dom} f_2 \subseteq [:X, Y:]$ and curry′ $f_1 =$ curry′ f_2, then $f_1 = f_2$.

(61) If $\operatorname{rng} f_1 \subseteq X \dot{\to} Y$ and $\operatorname{rng} f_2 \subseteq X \dot{\to} Y$ and $\emptyset \notin \operatorname{rng} f_1$ and $\emptyset \notin \operatorname{rng} f_2$ and uncurry $f_1 =$ uncurry f_2, then $f_1 = f_2$.

(62) If $\operatorname{rng} f_1 \subseteq X \dot{\to} Y$ and $\operatorname{rng} f_2 \subseteq X \dot{\to} Y$ and $\emptyset \notin \operatorname{rng} f_1$ and $\emptyset \notin \operatorname{rng} f_2$ and $\operatorname{uncurry}' f_1 = \operatorname{uncurry}' f_2$, then $f_1 = f_2$.

(63) If $X \subseteq Y$, then $X^Z \subseteq Y^Z$.

(64) $X^{\emptyset} = \{\emptyset\}$.

(65) $X \approx X^{\{x\}}$ and $\overline{\overline{X}} = \overline{\overline{X^{\{x\}}}}$.

(66) $\{x\}^X = \{X \longmapsto x\}$.

(67) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$, then ${X_2}^{X_1} \approx {Y_2}^{Y_1}$ and $\overline{\overline{{X_2}^{X_1}}} = \overline{\overline{{Y_2}^{Y_1}}}$.

(68) If $\overline{\overline{X_1}} = \overline{\overline{Y_1}}$ and $\overline{\overline{X_2}} = \overline{\overline{Y_2}}$, then $\overline{\overline{{X_2}^{X_1}}} = \overline{\overline{{Y_2}^{Y_1}}}$.

(69) If X_1 misses X_2, then $X^{X_1 \cup X_2} \approx [: X^{X_1}, X^{X_2} :]$ and $\overline{\overline{X^{X_1 \cup X_2}}} = \overline{\overline{[: X^{X_1}, X^{X_2} :]}}$.

(70) $Z^{[: X, Y :]} \approx (Z^Y)^X$ and $\overline{\overline{Z^{[: X, Y :]}}} = \overline{\overline{(Z^Y)^X}}$.

(71) $[: X, Y :]^Z \approx [: X^Z, Y^Z :]$ and $\overline{\overline{[: X, Y :]^Z}} = \overline{\overline{[: X^Z, Y^Z :]}}$.

(72) If $x \neq y$, then $\{x, y\}^X \approx 2^X$ and $\overline{\overline{\{x, y\}^X}} = \overline{\overline{2^X}}$.

(73) If $x \neq y$, then $X^{\{x, y\}} \approx [: X, X :]$ and $\overline{\overline{X^{\{x, y\}}}} = \overline{\overline{[: X, X :]}}$.

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