JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Functions and Their Basic Properties

Czesław Byliński
Warsaw University
Białystok

Summary. The definitions of the mode Function and the graph of a function are introduced. The graph of a function is defined to be identical with the function. The following concepts are also defined: the domain of a function, the range of a function, the identity function, the composition of functions, the 1-1 function, the inverse function, the restriction of a function, the image and the inverse image. Certain basic facts about functions and the notions defined in the article are proved.

MML Identifier: `FUNCT_1`.

WWW: `http://mizar.org/JFM/Vol1/funct_1.html`

The articles [1] and [2] provide the notation and terminology for this paper.

We use the following convention: X, X_1, X_2, Y, Y_1, Y_2 are sets and p, x, x_1, x_2, y, y_1, y_2, z are sets.

Let X be a set. We say that X is function-like if and only if:

(Def. 1) For all x, y_1, y_2 such that $\langle x, y_1 \rangle \in X$ and $\langle x, y_2 \rangle \in X$ holds $y_1 = y_2$.

Let us observe that there exists a set which is relation-like and function-like.

A function is a function-like relation-like set.

One can check that every set which is empty is also function-like.

We follow the rules: f, g, h denote functions and R, S denote binary relations.

Next we state the proposition

(2)[1] Let F be a set. Suppose for every p such that $p \in F$ there exist x, y such that $\langle x, y \rangle = p$ and for all x, y_1, y_2 such that $\langle x, y_1 \rangle \in F$ and $\langle x, y_2 \rangle \in F$ holds $y_1 = y_2$. Then F is a function.

The scheme *GraphFunc* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists f such that for all x, y holds $\langle x, y \rangle \in f$ iff $x \in \mathcal{A}$ and $\mathcal{P}[x, y]$

provided the parameters meet the following requirement:

- For all x, y_1, y_2 such that $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$.

Let us consider f, x. The functor $f(x)$ yielding a set is defined as follows:

(Def. 4)[2] (i) $\langle x, f(x) \rangle \in f$ if $x \in \operatorname{dom} f$,

(ii) $f(x) = \emptyset$, otherwise.

One can prove the following propositions:

(8)[3] $\langle x, y \rangle \in f$ iff $x \in \operatorname{dom} f$ and $y = f(x)$.

[1] The proposition (1) has been removed.
[2] The definitions (Def. 2) and (Def. 3) have been removed.
[3] The propositions (3)–(7) have been removed.

(9) If $\operatorname{dom} f = \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$, then $f = g$.

Let us consider f. Then $\operatorname{rng} f$ can be characterized by the condition:

(Def. 5) For every y holds $y \in \operatorname{rng} f$ iff there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$.

We now state two propositions:

(12)[4] If $x \in \operatorname{dom} f$, then $f(x) \in \operatorname{rng} f$.

(14)[5] If $\operatorname{dom} f = \{x\}$, then $\operatorname{rng} f = \{f(x)\}$.

Now we present two schemes. The scheme *FuncEx* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters meet the following requirements:

- For all x, y_1, y_2 such that $x \in \mathcal{A}$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$, and
- For every x such that $x \in \mathcal{A}$ there exists y such that $\mathcal{P}[x, y]$.

The scheme *Lambda* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

One can prove the following propositions:

(15) If $X \neq \emptyset$, then for every y there exists f such that $\operatorname{dom} f = X$ and $\operatorname{rng} f = \{y\}$.

(16) If for all f, g such that $\operatorname{dom} f = X$ and $\operatorname{dom} g = X$ holds $f = g$, then $X = \emptyset$.

(17) If $\operatorname{dom} f = \operatorname{dom} g$ and $\operatorname{rng} f = \{y\}$ and $\operatorname{rng} g = \{y\}$, then $f = g$.

(18) If $Y \neq \emptyset$ or $X = \emptyset$, then there exists f such that $X = \operatorname{dom} f$ and $\operatorname{rng} f \subseteq Y$.

(19) If for every y such that $y \in Y$ there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$, then $Y \subseteq \operatorname{rng} f$.

Let us consider f, g. We introduce $g \cdot f$ as a synonym of $f \cdot g$.

Let us consider f, g. One can check that $g \cdot f$ is function-like.

We now state several propositions:

(20) Let given h. Suppose for every x holds $x \in \operatorname{dom} h$ iff $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} h$ holds $h(x) = g(f(x))$. Then $h = g \cdot f$.

(21) $x \in \operatorname{dom}(g \cdot f)$ iff $x \in \operatorname{dom} f$ and $f(x) \in \operatorname{dom} g$.

(22) If $x \in \operatorname{dom}(g \cdot f)$, then $(g \cdot f)(x) = g(f(x))$.

(23) If $x \in \operatorname{dom} f$, then $(g \cdot f)(x) = g(f(x))$.

(25)[6] If $z \in \operatorname{rng}(g \cdot f)$, then $z \in \operatorname{rng} g$.

(27)[7] If $\operatorname{dom}(g \cdot f) = \operatorname{dom} f$, then $\operatorname{rng} f \subseteq \operatorname{dom} g$.

(33)[8] If $\operatorname{rng} f \subseteq Y$ and for all g, h such that $\operatorname{dom} g = Y$ and $\operatorname{dom} h = Y$ and $g \cdot f = h \cdot f$ holds $g = h$, then $Y = \operatorname{rng} f$.

Let us consider X. One can check that id_X is function-like.

Next we state several propositions:

---

[4] The propositions (10) and (11) have been removed.
[5] The proposition (13) has been removed.
[6] The proposition (24) has been removed.
[7] The proposition (26) has been removed.
[8] The propositions (28)–(32) have been removed.

(34) $f = \mathrm{id}_X$ iff $\mathrm{dom}\, f = X$ and for every x such that $x \in X$ holds $f(x) = x$.

(35) If $x \in X$, then $\mathrm{id}_X(x) = x$.

(37)[9] $\mathrm{dom}(f \cdot \mathrm{id}_X) = \mathrm{dom}\, f \cap X$.

(38) If $x \in \mathrm{dom}\, f \cap X$, then $f(x) = (f \cdot \mathrm{id}_X)(x)$.

(40)[10] $x \in \mathrm{dom}(\mathrm{id}_Y \cdot f)$ iff $x \in \mathrm{dom}\, f$ and $f(x) \in Y$.

(42)[11] $f \cdot \mathrm{id}_{\mathrm{dom}\, f} = f$ and $\mathrm{id}_{\mathrm{rng}\, f} \cdot f = f$.

(43) $\mathrm{id}_X \cdot \mathrm{id}_Y = \mathrm{id}_{X \cap Y}$.

(44) If $\mathrm{rng}\, f = \mathrm{dom}\, g$ and $g \cdot f = f$, then $g = \mathrm{id}_{\mathrm{dom}\, g}$.

Let us consider f. We say that f is one-to-one if and only if:

(Def. 8)[12] For all x_1, x_2 such that $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$.

Next we state several propositions:

(46)[13] If f is one-to-one and g is one-to-one, then $g \cdot f$ is one-to-one.

(47) If $g \cdot f$ is one-to-one and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, g$, then f is one-to-one.

(48) If $g \cdot f$ is one-to-one and $\mathrm{rng}\, f = \mathrm{dom}\, g$, then f is one-to-one and g is one-to-one.

(49) f is one-to-one iff for all g, h such that $\mathrm{rng}\, g \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}\, h \subseteq \mathrm{dom}\, f$ and $\mathrm{dom}\, g = \mathrm{dom}\, h$ and $f \cdot g = f \cdot h$ holds $g = h$.

(50) If $\mathrm{dom}\, f = X$ and $\mathrm{dom}\, g = X$ and $\mathrm{rng}\, g \subseteq X$ and f is one-to-one and $f \cdot g = f$, then $g = \mathrm{id}_X$.

(51) If $\mathrm{rng}(g \cdot f) = \mathrm{rng}\, g$ and g is one-to-one, then $\mathrm{dom}\, g \subseteq \mathrm{rng}\, f$.

(52) id_X is one-to-one.

(53) If there exists g such that $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$, then f is one-to-one.

One can verify that there exists a function which is empty.
Let us observe that every function which is empty is also one-to-one.
One can verify that there exists a function which is one-to-one.
Let f be an one-to-one function. Observe that $f^{\smile}$ is function-like.
Let us consider f. Let us assume that f is one-to-one. The functor f^{-1} yields a function and is defined by:

(Def. 9) $f^{-1} = f^{\smile}$.

The following propositions are true:

(54) Suppose f is one-to-one. Let g be a function. Then $g = f^{-1}$ if and only if the following conditions are satisfied:

(i) $\mathrm{dom}\, g = \mathrm{rng}\, f$, and

(ii) for all y, x holds $y \in \mathrm{rng}\, f$ and $x = g(y)$ iff $x \in \mathrm{dom}\, f$ and $y = f(x)$.

(55) If f is one-to-one, then $\mathrm{rng}\, f = \mathrm{dom}(f^{-1})$ and $\mathrm{dom}\, f = \mathrm{rng}(f^{-1})$.

(56) If f is one-to-one and $x \in \mathrm{dom}\, f$, then $x = f^{-1}(f(x))$ and $x = (f^{-1} \cdot f)(x)$.

---

[9] The proposition (36) has been removed.
[10] The proposition (39) has been removed.
[11] The proposition (41) has been removed.
[12] The definitions (Def. 6) and (Def. 7) have been removed.
[13] The proposition (45) has been removed.

(57) If f is one-to-one and $y \in \operatorname{rng} f$, then $y = f(f^{-1}(y))$ and $y = (f \cdot f^{-1})(y)$.

(58) If f is one-to-one, then $\operatorname{dom}(f^{-1} \cdot f) = \operatorname{dom} f$ and $\operatorname{rng}(f^{-1} \cdot f) = \operatorname{dom} f$.

(59) If f is one-to-one, then $\operatorname{dom}(f \cdot f^{-1}) = \operatorname{rng} f$ and $\operatorname{rng}(f \cdot f^{-1}) = \operatorname{rng} f$.

(60) Suppose f is one-to-one and $\operatorname{dom} f = \operatorname{rng} g$ and $\operatorname{rng} f = \operatorname{dom} g$ and for all x, y such that $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} g$ holds $f(x) = y$ iff $g(y) = x$. Then $g = f^{-1}$.

(61) If f is one-to-one, then $f^{-1} \cdot f = \mathrm{id}_{\operatorname{dom} f}$ and $f \cdot f^{-1} = \mathrm{id}_{\operatorname{rng} f}$.

(62) If f is one-to-one, then f^{-1} is one-to-one.

(63) If f is one-to-one and $\operatorname{rng} f = \operatorname{dom} g$ and $g \cdot f = \mathrm{id}_{\operatorname{dom} f}$, then $g = f^{-1}$.

(64) If f is one-to-one and $\operatorname{rng} g = \operatorname{dom} f$ and $f \cdot g = \mathrm{id}_{\operatorname{rng} f}$, then $g = f^{-1}$.

(65) If f is one-to-one, then $(f^{-1})^{-1} = f$.

(66) If f is one-to-one and g is one-to-one, then $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$.

(67) $(\mathrm{id}_X)^{-1} = \mathrm{id}_X$.

Let us consider f, X. One can verify that $f{\restriction}X$ is function-like.
One can prove the following propositions:

(68) $g = f{\restriction}X$ iff $\operatorname{dom} g = \operatorname{dom} f \cap X$ and for every x such that $x \in \operatorname{dom} g$ holds $g(x) = f(x)$.

(70)[14] If $x \in \operatorname{dom}(f{\restriction}X)$, then $(f{\restriction}X)(x) = f(x)$.

(71) If $x \in \operatorname{dom} f \cap X$, then $(f{\restriction}X)(x) = f(x)$.

(72) If $x \in X$, then $(f{\restriction}X)(x) = f(x)$.

(73) If $x \in \operatorname{dom} f$ and $x \in X$, then $f(x) \in \operatorname{rng}(f{\restriction}X)$.

(76)[15] $\operatorname{dom}(f{\restriction}X) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(f{\restriction}X) \subseteq \operatorname{rng} f$.

(82)[16] If $X \subseteq Y$, then $f{\restriction}X{\restriction}Y = f{\restriction}X$ and $f{\restriction}Y{\restriction}X = f{\restriction}X$.

(84)[17] If f is one-to-one, then $f{\restriction}X$ is one-to-one.

Let us consider Y, f. Observe that $Y{\upharpoonright}f$ is function-like.
One can prove the following propositions:

(85) $g = Y{\upharpoonright}f$ if and only if the following conditions are satisfied:
 (i) for every x holds $x \in \operatorname{dom} g$ iff $x \in \operatorname{dom} f$ and $f(x) \in Y$, and
 (ii) for every x such that $x \in \operatorname{dom} g$ holds $g(x) = f(x)$.

(86) $x \in \operatorname{dom}(Y{\upharpoonright}f)$ iff $x \in \operatorname{dom} f$ and $f(x) \in Y$.

(87) If $x \in \operatorname{dom}(Y{\upharpoonright}f)$, then $(Y{\upharpoonright}f)(x) = f(x)$.

(89)[18] $\operatorname{dom}(Y{\upharpoonright}f) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(Y{\upharpoonright}f) \subseteq \operatorname{rng} f$.

(97)[19] If $X \subseteq Y$, then $Y{\upharpoonright}(X{\upharpoonright}f) = X{\upharpoonright}f$ and $X{\upharpoonright}(Y{\upharpoonright}f) = X{\upharpoonright}f$.

[14] The proposition (69) has been removed.
[15] The propositions (74) and (75) have been removed.
[16] The propositions (77)–(81) have been removed.
[17] The proposition (83) has been removed.
[18] The proposition (88) has been removed.
[19] The propositions (90)–(96) have been removed.

(99)[20] If f is one-to-one, then $Y \upharpoonright f$ is one-to-one.

Let us consider f, X. Then $f^{\circ}X$ can be characterized by the condition:

(Def. 12)[21] For every y holds $y \in f^{\circ}X$ iff there exists x such that $x \in \mathrm{dom}\, f$ and $x \in X$ and $y = f(x)$.

One can prove the following propositions:

(117)[22] If $x \in \mathrm{dom}\, f$, then $f^{\circ}\{x\} = \{f(x)\}$.

(118) If $x_1 \in \mathrm{dom}\, f$ and $x_2 \in \mathrm{dom}\, f$, then $f^{\circ}\{x_1, x_2\} = \{f(x_1), f(x_2)\}$.

(120)[23] $(Y \upharpoonright f)^{\circ}X \subseteq f^{\circ}X$.

(121) If f is one-to-one, then $f^{\circ}(X_1 \cap X_2) = f^{\circ}X_1 \cap f^{\circ}X_2$.

(122) If for all X_1, X_2 holds $f^{\circ}(X_1 \cap X_2) = f^{\circ}X_1 \cap f^{\circ}X_2$, then f is one-to-one.

(123) If f is one-to-one, then $f^{\circ}(X_1 \setminus X_2) = f^{\circ}X_1 \setminus f^{\circ}X_2$.

(124) If for all X_1, X_2 holds $f^{\circ}(X_1 \setminus X_2) = f^{\circ}X_1 \setminus f^{\circ}X_2$, then f is one-to-one.

(125) If X misses Y and f is one-to-one, then $f^{\circ}X$ misses $f^{\circ}Y$.

(126) $(Y \upharpoonright f)^{\circ}X = Y \cap f^{\circ}X$.

Let us consider f, Y. Then $f^{-1}(Y)$ can be characterized by the condition:

(Def. 13) For every x holds $x \in f^{-1}(Y)$ iff $x \in \mathrm{dom}\, f$ and $f(x) \in Y$.

We now state a number of propositions:

(137)[24] $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

(138) $f^{-1}(Y_1 \setminus Y_2) = f^{-1}(Y_1) \setminus f^{-1}(Y_2)$.

(139) $(R \upharpoonright X)^{-1}(Y) = X \cap R^{-1}(Y)$.

(142)[25] $y \in \mathrm{rng}\, R$ iff $R^{-1}(\{y\}) \neq \emptyset$.

(143) If for every y such that $y \in Y$ holds $R^{-1}(\{y\}) \neq \emptyset$, then $Y \subseteq \mathrm{rng}\, R$.

(144) For every y such that $y \in \mathrm{rng}\, f$ there exists x such that $f^{-1}(\{y\}) = \{x\}$ iff f is one-to-one.

(145) $f^{\circ}f^{-1}(Y) \subseteq Y$.

(146) If $X \subseteq \mathrm{dom}\, R$, then $X \subseteq R^{-1}(R^{\circ}X)$.

(147) If $Y \subseteq \mathrm{rng}\, f$, then $f^{\circ}f^{-1}(Y) = Y$.

(148) $f^{\circ}f^{-1}(Y) = Y \cap f^{\circ}\mathrm{dom}\, f$.

(149) $f^{\circ}(X \cap f^{-1}(Y)) \subseteq f^{\circ}X \cap Y$.

(150) $f^{\circ}(X \cap f^{-1}(Y)) = f^{\circ}X \cap Y$.

(151) $X \cap R^{-1}(Y) \subseteq R^{-1}(R^{\circ}X \cap Y)$.

(152) If f is one-to-one, then $f^{-1}(f^{\circ}X) \subseteq X$.

[20] The proposition (98) has been removed.
[21] The definitions (Def. 10) and (Def. 11) have been removed.
[22] The propositions (100)–(116) have been removed.
[23] The proposition (119) has been removed.
[24] The propositions (127)–(136) have been removed.
[25] The propositions (140) and (141) have been removed.

(153) If for every X holds $f^{-1}(f^{\circ}X) \subseteq X$, then f is one-to-one.

(154) If f is one-to-one, then $f^{\circ}X = (f^{-1})^{-1}(X)$.

(155) If f is one-to-one, then $f^{-1}(Y) = (f^{-1})^{\circ}Y$.

(156) If $Y = \operatorname{rng} f$ and $\operatorname{dom} g = Y$ and $\operatorname{dom} h = Y$ and $g \cdot f = h \cdot f$, then $g = h$.

(157) If $f^{\circ}X_1 \subseteq f^{\circ}X_2$ and $X_1 \subseteq \operatorname{dom} f$ and f is one-to-one, then $X_1 \subseteq X_2$.

(158) If $f^{-1}(Y_1) \subseteq f^{-1}(Y_2)$ and $Y_1 \subseteq \operatorname{rng} f$, then $Y_1 \subseteq Y_2$.

(159) f is one-to-one iff for every y there exists x such that $f^{-1}(\{y\}) \subseteq \{x\}$.

(160) If $\operatorname{rng} R \subseteq \operatorname{dom} S$, then $R^{-1}(X) \subseteq (R \cdot S)^{-1}(S^{\circ}X)$.

REFERENCES

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[2] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received March 3, 1989

Published January 2, 2004