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Binary Operations on Finite Sequences

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Summary. We generalize the semigroup operation on finite sequences introduced in [8] for binary operations that have a unity or for non-empty finite sequences.

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The articles [11], [15], [12], [3], [6], [2], [14], [16], [17], [4], [13], [10], [5], [1], [9], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: D is a non empty set, d, d_1, d_2, d_3 are elements of D, F, G, H are finite sequences of elements of D, f is a function from $\mathbb{N}$ into D, g is a binary operation on D, k, n, l are natural numbers, and P is a permutation of $\mathrm{dom}F$.

Let us consider D, n, d. Then $n \mapsto d$ is a finite sequence of elements of D.

Let us consider D, F, g. Let us assume that g has a unity or $\mathrm{len}F \geq 1$. The functor $g \odot F$ yields an element of D and is defined by:

(Def. 1)(i) $g \odot F = \mathbf{1}_g$ if g has a unity and $\mathrm{len}F = 0$,

(ii) there exists f such that $f(1) = F(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}F$ holds $f(n+1) = g(f(n), F(n+1))$ and $g \odot F = f(\mathrm{len}F)$, otherwise.

One can prove the following propositions:

(2)[1] If $\mathrm{len}F \geq 1$, then there exists f such that $f(1) = F(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}F$ holds $f(n+1) = g(f(n), F(n+1))$ and $g \odot F = f(\mathrm{len}F)$.

(3) Suppose $\mathrm{len}F \geq 1$ and there exists f such that $f(1) = F(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}F$ holds $f(n+1) = g(f(n), F(n+1))$ and $d = f(\mathrm{len}F)$. Then $d = g \odot F$.

Let B, A be non empty sets and let b be an element of B. Then $A \longmapsto b$ is a function from A into B.

Let A be a non empty set, let F be a function from $\mathbb{N}$ into A, and let p be a finite sequence of elements of A. Then $F {+\cdot} p$ is a function from $\mathbb{N}$ into A.

Let f be a finite sequence. Then $\mathrm{dom} f$ is an element of $\mathrm{Fin}\,\mathbb{N}$.

The following propositions are true:

(4) If g has a unity or $\mathrm{len}F \geq 1$ and if g is associative and commutative, then $g \odot F = g\text{-}\sum_{\mathrm{dom}F}((\mathbb{N} \longmapsto \mathbf{1}_g) {+\cdot} F)$.

(5) If g has a unity or $\mathrm{len}F \geq 1$, then $g \odot F \frown \langle d \rangle = g(g \odot F, d)$.

(6) If g is associative and if g has a unity or $\mathrm{len}F \geq 1$ and $\mathrm{len}G \geq 1$, then $g \odot F \frown G = g(g \odot F, g \odot G)$.

[1] The proposition (1) has been removed.

(7) If g is associative and if g has a unity or $\operatorname{len} F \geq 1$, then $g \odot \langle d \rangle \frown F = g(d, g \odot F)$.

(8) If g is commutative and associative and if g has a unity or $\operatorname{len} F \geq 1$ and if $G = F \cdot P$, then $g \odot F = g \odot G$.

(9) Suppose g has a unity or $\operatorname{len} F \geq 1$ and g is associative and commutative and F is one-to-one and G is one-to-one and $\operatorname{rng} F = \operatorname{rng} G$. Then $g \odot F = g \odot G$.

(10) Suppose that

(i) g is associative and commutative,

(ii) g has a unity or $\operatorname{len} F \geq 1$,

(iii) $\operatorname{len} F = \operatorname{len} G$,

(iv) $\operatorname{len} F = \operatorname{len} H$, and

(v) for every k such that $k \in \operatorname{dom} F$ holds $F(k) = g(G(k), H(k))$.

Then $g \odot F = g(g \odot G, g \odot H)$.

(11) If g has a unity, then $g \odot \varepsilon_D = \mathbf{1}_g$.

(12) $g \odot \langle d \rangle = d$.

(13) $g \odot \langle d_1, d_2 \rangle = g(d_1, d_2)$.

(14) If g is commutative, then $g \odot \langle d_1, d_2 \rangle = g \odot \langle d_2, d_1 \rangle$.

(15) $g \odot \langle d_1, d_2, d_3 \rangle = g(g(d_1, d_2), d_3)$.

(16) If g is commutative, then $g \odot \langle d_1, d_2, d_3 \rangle = g \odot \langle d_2, d_1, d_3 \rangle$.

(17) $g \odot 1 \mapsto d = d$.

(18) $g \odot 2 \mapsto d = g(d, d)$.

(19) If g is associative and if g has a unity or $k \neq 0$ and $l \neq 0$, then $g \odot (k+l) \mapsto d = g(g \odot k \mapsto d, g \odot l \mapsto d)$.

(20) If g is associative and if g has a unity or $k \neq 0$ and $l \neq 0$, then $g \odot (k \cdot l) \mapsto d = g \odot l \mapsto (g \odot k \mapsto d)$.

(21) If $\operatorname{len} F = 1$, then $g \odot F = F(1)$.

(22) If $\operatorname{len} F = 2$, then $g \odot F = g(F(1), F(2))$.

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