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Pigeon Hole Principle

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Summary. We introduce the notion of a predicate that states that a function is one-to-one at a given element of its domain (i.e. counterimage of image of the element is equal to its singleton). We also introduce some rather technical functors concerning finite sequences: the lowest index of the given element of the range of the finite sequence, the substring preceding (and succeeding) the first occurrence of given element of the range. At the end of the article we prove the pigeon hole principle.

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The articles [7], [10], [8], [3], [11], [4], [1], [5], [6], [2], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: f is a function, p, q are finite sequences, A, B, x, y, z are sets, and i, k, n are natural numbers.

Let us consider f, x. We say that f is one-to-one at x if and only if:

(Def. 1) $f^{-1}(f^{\circ}\{x\}) = \{x\}$.

The following propositions are true:

(2)[1] If f is one-to-one at x, then $x \in \operatorname{dom} f$.

(3) f is one-to-one at x iff $x \in \operatorname{dom} f$ and $f^{-1}(\{f(x)\}) = \{x\}$.

(4) f is one-to-one at x iff $x \in \operatorname{dom} f$ and for every z such that $z \in \operatorname{dom} f$ and $x \neq z$ holds $f(x) \neq f(z)$.

(5) For every x such that $x \in \operatorname{dom} f$ holds f is one-to-one at x iff f is one-to-one.

Let us consider f, y. We say that f yields y just once if and only if:

(Def. 2) There exists a finite set B such that $B = f^{-1}(\{y\})$ and $\operatorname{card} B = 1$.

We now state several propositions:

(7)[2] If f yields y just once, then $y \in \operatorname{rng} f$.

(8) f yields y just once iff there exists x such that $\{x\} = f^{-1}(\{y\})$.

(9) f yields y just once if and only if there exists x such that $x \in \operatorname{dom} f$ and $y = f(x)$ and for every z such that $z \in \operatorname{dom} f$ and $z \neq x$ holds $f(z) \neq y$.

[1] The proposition (1) has been removed.
[2] The proposition (6) has been removed.

(10) f is one-to-one iff for every y such that $y \in \operatorname{rng} f$ holds f yields y just once.

(11) f is one-to-one at x iff $x \in \operatorname{dom} f$ and f yields $f(x)$ just once.

Let us consider f, y. Let us assume that f yields y just once. The functor $f^{-1}(y)$ yielding a set is defined by:

(Def. 3) $f^{-1}(y) \in \operatorname{dom} f$ and $f(f^{-1}(y)) = y$.

We now state several propositions:

(16)[3] If f yields y just once, then $f^{\circ}\{f^{-1}(y)\} = \{y\}$.

(17) If f yields y just once, then $f^{-1}(\{y\}) = \{f^{-1}(y)\}$.

(18) If f is one-to-one and $y \in \operatorname{rng} f$, then $f^{-1}(y) = f^{-1}(y)$.

(20)[4] If f is one-to-one at x, then $f^{-1}(f(x)) = x$.

(21) If f yields y just once, then f is one-to-one at $f^{-1}(y)$.

We use the following convention: D denotes a non empty set and d, d_1, d_2, d_3 denote elements of D.

Let us consider D and let us consider d_1, d_2. Then $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

Let us consider D and let us consider d_1, d_2, d_3. Then $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

Let X, D be sets, let p be a partial function from X to D, and let i be a set. Let us assume that $i \in \operatorname{dom} p$. The functor p_i yields an element of D and is defined as follows:

(Def. 4) $p_i = p(i)$.

The following propositions are true:

(22) For all non empty sets X, D and for every function p from X into D and for every element i of X holds $p_i = p(i)$.

(24)[5] For every set D and for every finite sequence P of elements of D and for every i such that $1 \leq i$ and $i \leq \operatorname{len} P$ holds $P_i = P(i)$.

(25) $\langle d \rangle_1 = d$.

(26) $\langle d_1, d_2 \rangle_1 = d_1$ and $\langle d_1, d_2 \rangle_2 = d_2$.

(27) $\langle d_1, d_2, d_3 \rangle_1 = d_1$ and $\langle d_1, d_2, d_3 \rangle_2 = d_2$ and $\langle d_1, d_2, d_3 \rangle_3 = d_3$.

Let us consider p and let us consider x. The functor $x \hookleftarrow p$ yielding a natural number is defined as follows:

(Def. 5) $x \hookleftarrow p = (\operatorname{Sgm}(p^{-1}(\{x\})))(1)$.

Next we state a number of propositions:

(29)[6] If $x \in \operatorname{rng} p$, then $p(x \hookleftarrow p) = x$.

(30) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p \in \operatorname{dom} p$.

(31) If $x \in \operatorname{rng} p$, then $1 \leq x \hookleftarrow p$ and $x \hookleftarrow p \leq \operatorname{len} p$.

(32) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p - 1$ is a natural number and $\operatorname{len} p - x \hookleftarrow p$ is a natural number.

[3] The propositions (12)–(15) have been removed.
[4] The proposition (19) has been removed.
[5] The proposition (23) has been removed.
[6] The proposition (28) has been removed.

(33) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p \in p^{-1}(\{x\})$.

(34) For every k such that $k \in \operatorname{dom} p$ and $k < x \hookleftarrow p$ holds $p(k) \neq x$.

(35) If p yields x just once, then $p^{-1}(x) = x \hookleftarrow p$.

(36) If p yields x just once, then for every k such that $k \in \operatorname{dom} p$ and $k \neq x \hookleftarrow p$ holds $p(k) \neq x$.

(37) If $x \in \operatorname{rng} p$ and for every k such that $k \in \operatorname{dom} p$ and $k \neq x \hookleftarrow p$ holds $p(k) \neq x$, then p yields x just once.

(38) p yields x just once iff $x \in \operatorname{rng} p$ and $\{x \hookleftarrow p\} = p^{-1}(\{x\})$.

(39) If p is one-to-one and $x \in \operatorname{rng} p$, then $\{x \hookleftarrow p\} = p^{-1}(\{x\})$.

(40) p yields x just once iff $\operatorname{len}(p - \{x\}) = \operatorname{len} p - 1$.

(41) Suppose p yields x just once. Let given k such that $k \in \operatorname{dom}(p - \{x\})$. Then

(i) if $k < x \hookleftarrow p$, then $(p - \{x\})(k) = p(k)$, and

(ii) if $x \hookleftarrow p \leq k$, then $(p - \{x\})(k) = p(k+1)$.

(42) Suppose p is one-to-one and $x \in \operatorname{rng} p$. Let given k such that $k \in \operatorname{dom}(p - \{x\})$. Then

(i) $(p - \{x\})(k) = p(k)$ iff $k < x \hookleftarrow p$, and

(ii) $(p - \{x\})(k) = p(k+1)$ iff $x \hookleftarrow p \leq k$.

Let us consider p and let us consider x. Let us assume that $x \in \operatorname{rng} p$. The functor $p \leftarrow x$ yields a finite sequence and is defined by:

(Def. 6) There exists n such that $n = x \hookleftarrow p - 1$ and $p \leftarrow x = p \restriction \operatorname{Seg} n$.

Next we state several propositions:

(45)[7] If $x \in \operatorname{rng} p$ and $n = x \hookleftarrow p - 1$, then $p \restriction \operatorname{Seg} n = p \leftarrow x$.

(46) If $x \in \operatorname{rng} p$, then $\operatorname{len}(p \leftarrow x) = x \hookleftarrow p - 1$.

(47) If $x \in \operatorname{rng} p$ and $n = x \hookleftarrow p - 1$, then $\operatorname{dom}(p \leftarrow x) = \operatorname{Seg} n$.

(48) If $x \in \operatorname{rng} p$ and $k \in \operatorname{dom}(p \leftarrow x)$, then $p(k) = (p \leftarrow x)(k)$.

(49) If $x \in \operatorname{rng} p$, then $x \notin \operatorname{rng}(p \leftarrow x)$.

(50) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \leftarrow x)$ misses $\{x\}$.

(51) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \leftarrow x) \subseteq \operatorname{rng} p$.

(52) If $x \in \operatorname{rng} p$, then $x \hookleftarrow p = 1$ iff $p \leftarrow x = \emptyset$.

(53) If $x \in \operatorname{rng} p$ and p is a finite sequence of elements of D, then $p \leftarrow x$ is a finite sequence of elements of D.

Let us consider p and let us consider x. Let us assume that $x \in \operatorname{rng} p$. The functor $p \rightarrow x$ yielding a finite sequence is defined by:

(Def. 7) $\operatorname{len}(p \rightarrow x) = \operatorname{len} p - x \hookleftarrow p$ and for every k such that $k \in \operatorname{dom}(p \rightarrow x)$ holds $(p \rightarrow x)(k) = p(k + x \hookleftarrow p)$.

One can prove the following propositions:

(57)[8] If $x \in \operatorname{rng} p$ and $n = \operatorname{len} p - x \hookleftarrow p$, then $\operatorname{dom}(p \rightarrow x) = \operatorname{Seg} n$.

[7] The propositions (43) and (44) have been removed.

[8] The propositions (54)–(56) have been removed.

(58) If $x \in \operatorname{rng} p$ and $n \in \operatorname{dom}(p \rightarrow x)$, then $n + x \leftarrow\!\!\!\!\!\!\wp\ p \in \operatorname{dom} p$.

(59) If $x \in \operatorname{rng} p$, then $\operatorname{rng}(p \rightarrow x) \subseteq \operatorname{rng} p$.

(60) p yields x just once iff $x \in \operatorname{rng} p$ and $x \notin \operatorname{rng}(p \rightarrow x)$.

(61) If $x \in \operatorname{rng} p$ and p is one-to-one, then $x \notin \operatorname{rng}(p \rightarrow x)$.

(62) p yields x just once iff $x \in \operatorname{rng} p$ and $\operatorname{rng}(p \rightarrow x)$ misses $\{x\}$.

(63) If $x \in \operatorname{rng} p$ and p is one-to-one, then $\operatorname{rng}(p \rightarrow x)$ misses $\{x\}$.

(64) If $x \in \operatorname{rng} p$, then $x \leftarrow\!\!\!\!\!\!\wp\ p = \operatorname{len} p$ iff $p \rightarrow x = \emptyset$.

(65) If $x \in \operatorname{rng} p$ and p is a finite sequence of elements of D, then $p \rightarrow x$ is a finite sequence of elements of D.

(66) If $x \in \operatorname{rng} p$, then $p = (p \leftarrow x) ^\frown \langle x \rangle ^\frown (p \rightarrow x)$.

(67) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p \leftarrow x$ is one-to-one.

(68) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p \rightarrow x$ is one-to-one.

(69) p yields x just once iff $x \in \operatorname{rng} p$ and $p - \{x\} = (p \leftarrow x) ^\frown (p \rightarrow x)$.

(70) If $x \in \operatorname{rng} p$ and p is one-to-one, then $p - \{x\} = (p \leftarrow x) ^\frown (p \rightarrow x)$.

(71) If $x \in \operatorname{rng} p$ and $p - \{x\}$ is one-to-one and $p - \{x\} = (p \leftarrow x) ^\frown (p \rightarrow x)$, then p is one-to-one.

(72) If $x \in \operatorname{rng} p$ and p is one-to-one, then $\operatorname{rng}(p \leftarrow x)$ misses $\operatorname{rng}(p \rightarrow x)$.

(73) If A is finite, then there exists p such that $\operatorname{rng} p = A$ and p is one-to-one.

(74) If $\operatorname{rng} p \subseteq \operatorname{dom} p$ and p is one-to-one, then $\operatorname{rng} p = \operatorname{dom} p$.

(75) If $\operatorname{rng} p = \operatorname{dom} p$, then p is one-to-one.

(76) If $\operatorname{rng} p = \operatorname{rng} q$ and $\operatorname{len} p = \operatorname{len} q$ and q is one-to-one, then p is one-to-one.

(77) p is one-to-one iff $\operatorname{card} \operatorname{rng} p = \operatorname{len} p$.

In the sequel f denotes a function from A into B.
We now state four propositions:

(78) For all finite sets A, B and for every function f from A into B such that $\operatorname{card} A = \operatorname{card} B$ and f is one-to-one holds $\operatorname{rng} f = B$.

(79) For all finite sets A, B and for every function f from A into B such that $\operatorname{card} A = \operatorname{card} B$ and $\operatorname{rng} f = B$ holds f is one-to-one.

(80) If $\overline{\overline{B}} < \overline{\overline{A}}$ and $B \neq \emptyset$, then there exist x, y such that $x \in A$ and $y \in A$ and $x \neq y$ and $f(x) = f(y)$.

(81) If $\overline{\overline{A}} < \overline{\overline{B}}$, then there exists x such that $x \in B$ and for every y such that $y \in A$ holds $f(y) \neq x$.

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