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Real Function Differentiability[1]

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Summary. For a real valued function defined on its domain in real numbers the differentiability in a single point and on a subset of the domain is presented. The main elements of differential calculus are developed. The algebraic properties of differential real functions are shown.

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The articles [11], [13], [1], [12], [3], [6], [4], [5], [14], [2], [7], [8], [10], and [9] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: X denotes a set, x, x_0, r, p denote real numbers, n denotes a natural number, Y denotes a subset of $\mathbb{R}$, Z denotes an open subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

We now state the proposition

(1) For every r holds $r \in Y$ iff $r \in \mathbb{R}$ iff $Y = \mathbb{R}$.

Let I_1 be a sequence of real numbers. We say that I_1 is convergent to 0 if and only if:

(Def. 1) I_1 is non-zero and convergent and $\lim I_1 = 0$.

Let us observe that there exists a sequence of real numbers which is convergent to 0.

One can check that there exists a sequence of real numbers which is constant.

In the sequel h denotes a convergent to 0 sequence of real numbers and c denotes a constant sequence of real numbers.

Let I_1 be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that I_1 is rest-like if and only if:

(Def. 3)[1] I_1 is total and for every h holds $h^{-1}\ (I_1 \cdot h)$ is convergent and $\lim(h^{-1}\ (I_1 \cdot h)) = 0$.

One can check that there exists a partial function from $\mathbb{R}$ to $\mathbb{R}$ which is rest-like.

A rest is a rest-like partial function from $\mathbb{R}$ to $\mathbb{R}$.

Let I_1 be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that I_1 is linear if and only if:

(Def. 4) I_1 is total and there exists r such that for every p holds $I_1(p) = r \cdot p$.

Let us note that there exists a partial function from $\mathbb{R}$ to $\mathbb{R}$ which is linear.

A linear function is a linear partial function from $\mathbb{R}$ to $\mathbb{R}$.

We use the following convention: R, R_1, R_2 denote rests and L, L_1, L_2 denote linear functions.

We now state several propositions:

[1]Supported by RPBP.III-24.C8.
[1] The definition (Def. 2) has been removed.

(6)[2] For all L_1, L_2 holds $L_1 + L_2$ is a linear function and $L_1 - L_2$ is a linear function.

(7) For all r, L holds $r\,L$ is a linear function.

(8) For all R_1, R_2 holds $R_1 + R_2$ is a rest and $R_1 - R_2$ is a rest and $R_1\,R_2$ is a rest.

(9) For all r, R holds $r\,R$ is a rest.

(10) $L_1\,L_2$ is rest-like.

(11) $R\,L$ is a rest and $L\,R$ is a rest.

Let us consider f and let x_0 be a real number. We say that f is differentiable in x_0 if and only if:

(Def. 5) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Let us consider f and let x_0 be a real number. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yielding a real number is defined by the condition (Def. 6).

(Def. 6) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that $f'(x_0) = L(1)$ and for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Let us consider f, X. We say that f is differentiable on X if and only if:

(Def. 7) $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

One can prove the following propositions:

(15)[3] If f is differentiable on X, then X is a subset of $\mathbb{R}$.

(16) f is differentiable on Z iff $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(17) If f is differentiable on Y, then Y is open.

Let us consider f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yields a partial function from $\mathbb{R}$ to $\mathbb{R}$ and is defined by:

(Def. 8) $\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $f'_{\restriction X}(x) = f'(x)$.

We now state the proposition

(19)[4] Let given f, Z. Suppose $Z \subseteq \operatorname{dom} f$ and there exists r such that $\operatorname{rng} f = \{r\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = 0$.

Let us consider h, n. One can verify that $h \uparrow n$ is convergent to 0.
Let us consider c, n. Observe that $c \uparrow n$ is constant.
We now state a number of propositions:

(20) Let x_0 be a real number and N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Let given h, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,(f \cdot (h + c) - f \cdot c)$ is convergent and $f'(x_0) = \lim(h^{-1}\,(f \cdot (h + c) - f \cdot c))$.

(21) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(22) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

[2] The propositions (2)–(5) have been removed.
[3] The propositions (12)–(14) have been removed.
[4] The proposition (18) has been removed.

(23) For all r, f, x_0 such that f is differentiable in x_0 holds $r\,f$ is differentiable in x_0 and $(r\,f)'(x_0) = r \cdot f'(x_0)$.

(24) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1\,f_2$ is differentiable in x_0 and $(f_1\,f_2)'(x_0) = f_2(x_0) \cdot {f_1}'(x_0) + f_1(x_0) \cdot {f_2}'(x_0)$.

(25) For all f, Z such that $Z \subseteq \mathrm{dom}\, f$ and $f{\restriction}Z = \mathrm{id}_Z$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{{\restriction}Z}(x) = 1$.

(26) Let given f_1, f_2, Z. Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{{\restriction}Z}(x) = {f_1}'(x) + {f_2}'(x)$.

(27) Let given f_1, f_2, Z. Suppose $Z \subseteq \mathrm{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 - f_2)'_{{\restriction}Z}(x) = {f_1}'(x) - {f_2}'(x)$.

(28) Let given r, f, Z. Suppose $Z \subseteq \mathrm{dom}(r\,f)$ and f is differentiable on Z. Then $r\,f$ is differentiable on Z and for every x such that $x \in Z$ holds $(r\,f)'_{{\restriction}Z}(x) = r \cdot f'(x)$.

(29) Let given f_1, f_2, Z. Suppose $Z \subseteq \mathrm{dom}(f_1\,f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1\,f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1\,f_2)'_{{\restriction}Z}(x) = f_2(x) \cdot {f_1}'(x) + f_1(x) \cdot {f_2}'(x)$.

(30) If $Z \subseteq \mathrm{dom}\, f$ and f is a constant on Z, then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{{\restriction}Z}(x) = 0$.

(31) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + p$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{{\restriction}Z}(x) = r$.

(32) For every real number x_0 such that f is differentiable in x_0 holds f is continuous in x_0.

(33) If f is differentiable on X, then f is continuous on X.

(34) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(35) If f is differentiable in x_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

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