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Convergence and the Limit of Complex Sequences. Series

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The articles [16], [3], [15], [6], [7], [17], [4], [10], [8], [2], [9], [12], [14], [5], [1], [11], and [13] provide the notation and terminology for this paper.

1. PRELIMINARIES

For simplicity, we adopt the following convention: r_1, r_2, r_3 are sequences of real numbers, s_1, s_2, s_3 are complex sequences, k, n, m are natural numbers, and p, r are real numbers.

We now state three propositions:

(1) $(n+1)+0i \neq 0_{\mathbb{C}}$ and $0+(n+1)i \neq 0_{\mathbb{C}}$.

(2) If for every n holds $r_1(n)=0$, then for every m holds $(\sum_{\alpha=0}^{\kappa}|r_1|(\alpha))_{\kappa\in\mathbb{N}}(m)=0$.

(3) If for every n holds $r_1(n)=0$, then r_1 is absolutely summable.

Let us observe that there exists a sequence of real numbers which is absolutely summable.

Let us note that every sequence of real numbers which is summable is also convergent.

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The following propositions are true:

(4) Suppose r_1 is convergent. Let given p. Suppose $0<p$. Then there exists n such that for all natural numbers m, l such that $n \leq m$ and $n \leq l$ holds $|r_1(m)-r_1(l)|<p$.

(5) If for every n holds $r_1(n) \leq p$, then for all natural numbers n, l holds $(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n+l)-(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq p\cdot l$.

(6) If for every n holds $r_1(n) \leq p$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq p\cdot(n+1)$.

(7) If for every n such that $n \leq m$ holds $r_2(n) \leq p\cdot r_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(r_2)(\alpha))_{\kappa\in\mathbb{N}}(m) \leq p\cdot(\sum_{\alpha=0}^{\kappa}(r_3)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(8) Suppose that for every n such that $n \leq m$ holds $r_2(n) \leq p\cdot r_3(n)$. Let given n. Suppose $n \leq m$. Let l be a natural number. If $n+l \leq m$, then $(\sum_{\alpha=0}^{\kappa}(r_2)(\alpha))_{\kappa\in\mathbb{N}}(n+l)-(\sum_{\alpha=0}^{\kappa}(r_2)(\alpha))_{\kappa\in\mathbb{N}}(n) \leq p\cdot((\sum_{\alpha=0}^{\kappa}(r_3)(\alpha))_{\kappa\in\mathbb{N}}(n+l)-(\sum_{\alpha=0}^{\kappa}(r_3)(\alpha))_{\kappa\in\mathbb{N}}(n))$.

(9) If for every n holds $0 \leq r_1(n)$, then for all n, m such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(m)-(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(m)-(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$ and for every n holds $|(\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| = (\sum_{\alpha=0}^{\kappa}(r_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(10) If s_2 is convergent and s_3 is convergent and $\lim(s_2 - s_3) = 0_{\mathbb{C}}$, then $\lim s_2 = \lim s_3$.

2. The Operations on Complex Sequences

In the sequel z is an element of $\mathbb{C}$ and N_1 is an increasing sequence of naturals.

Let z be an element of $\mathbb{C}$. The functor $(z^\kappa)_{\kappa\in\mathbb{N}}$ yielding a complex sequence is defined by:

(Def. 1) $(z^\kappa)_{\kappa\in\mathbb{N}}(0) = 1_{\mathbb{C}}$ and for every n holds $(z^\kappa)_{\kappa\in\mathbb{N}}(n+1) = (z^\kappa)_{\kappa\in\mathbb{N}}(n)\cdot z$.

Let z be an element of $\mathbb{C}$ and let n be a natural number. The functor $z_{\mathbb{N}}^{n}$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 2) $z_{\mathbb{N}}^{n} = (z^\kappa)_{\kappa\in\mathbb{N}}(n)$.

One can prove the following proposition

(11) $z_{\mathbb{N}}^{0} = 1_{\mathbb{C}}$.

Let c be a complex sequence. The functor $\Re(c)$ yields a sequence of real numbers and is defined by:

(Def. 3) For every n holds $\Re(c)(n) = \Re(c(n))$.

Let c be a complex sequence. The functor $\Im(c)$ yielding a sequence of real numbers is defined as follows:

(Def. 4) For every n holds $\Im(c)(n) = \Im(c(n))$.

Next we state a number of propositions:

(12) $|z| \leq |\Re(z)| + |\Im(z)|$.

(13) $|\Re(z)| \leq |z|$ and $|\Im(z)| \leq |z|$.

(14) If $\Re(s_2) = \Re(s_3)$ and $\Im(s_2) = \Im(s_3)$, then $s_2 = s_3$.

(15) $\Re(s_2) + \Re(s_3) = \Re(s_2 + s_3)$ and $\Im(s_2) + \Im(s_3) = \Im(s_2 + s_3)$.

(16) $-\Re(s_1) = \Re(-s_1)$ and $-\Im(s_1) = \Im(-s_1)$.

(17) $r \cdot \Re(z) = \Re((r + 0i)\cdot z)$ and $r \cdot \Im(z) = \Im((r + 0i)\cdot z)$.

(18) $\Re(s_2) - \Re(s_3) = \Re(s_2 - s_3)$ and $\Im(s_2) - \Im(s_3) = \Im(s_2 - s_3)$.

(19) $r\,\Re(s_1) = \Re((r + 0i)\,s_1)$ and $r\,\Im(s_1) = \Im((r + 0i)\,s_1)$.

(20) $\Re(z\,s_1) = \Re(z)\,\Re(s_1) - \Im(z)\,\Im(s_1)$ and $\Im(z\,s_1) = \Re(z)\,\Im(s_1) + \Im(z)\,\Re(s_1)$.

(21) $\Re(s_2\,s_3) = \Re(s_2)\,\Re(s_3) - \Im(s_2)\,\Im(s_3)$ and $\Im(s_2\,s_3) = \Re(s_2)\,\Im(s_3) + \Im(s_2)\,\Re(s_3)$.

Let s_1 be a complex sequence and let N_1 be an increasing sequence of naturals. The functor $s_1\,N_1$ yields a complex sequence and is defined as follows:

(Def. 5) For every n holds $(s_1\,N_1)(n) = s_1(N_1(n))$.

One can prove the following proposition

(22) $\Re(s_1\,N_1) = \Re(s_1)\cdot N_1$ and $\Im(s_1\,N_1) = \Im(s_1)\cdot N_1$.

Let s_1 be a complex sequence and let k be a natural number. The functor $s_1 \uparrow k$ yields a complex sequence and is defined by:

(Def. 6) For every n holds $(s_1 \uparrow k)(n) = s_1(n+k)$.

One can prove the following proposition

(23) $\Re(s_1)\uparrow k = \Re(s_1 \uparrow k)$ and $\Im(s_1)\uparrow k = \Im(s_1 \uparrow k)$.

Let s_1 be a complex sequence. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yields a complex sequence and is defined by:

(Def. 7) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

Let s_1 be a complex sequence. The functor $\sum s_1$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 8) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

One can prove the following propositions:

(24) If for every n holds $s_1(n) = 0_{\mathbb{C}}$, then for every m holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) = 0_{\mathbb{C}}$.

(25) If for every n holds $s_1(n) = 0_{\mathbb{C}}$, then for every m holds $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(m) = 0$.

(26) $(\sum_{\alpha=0}^{\kappa}\Re(s_1)(\alpha))_{\kappa\in\mathbb{N}} = \Re((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$ and $(\sum_{\alpha=0}^{\kappa}\Im(s_1)(\alpha))_{\kappa\in\mathbb{N}} = \Im((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

(27) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2+s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(28) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2-s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(29) $(\sum_{\alpha=0}^{\kappa}(z\,s_1)(\alpha))_{\kappa\in\mathbb{N}} = z\,(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(30) $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(k)| \leq (\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(k)$.

(31) $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| \leq |(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(32) $(\sum_{\alpha=0}^{\kappa}\Re(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow k = \Re((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow k)$ and $(\sum_{\alpha=0}^{\kappa}\Im(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow k = \Im((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow k)$.

(33) If for every n holds $s_2(n) = s_1(0)$, then $(\sum_{\alpha=0}^{\kappa}(s_1\uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_2$.

(34) $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

Let s_1 be a complex sequence. Note that $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.
We now state three propositions:

(35) If for every n such that $n \leq m$ holds $s_2(n) = s_3(n)$, then $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}}(m) = (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(m)$.

(36) If $1_{\mathbb{C}} \neq z$, then for every n holds $(\sum_{\alpha=0}^{\kappa}((z^{\kappa})_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = \frac{1_{\mathbb{C}} - z_{\mathbb{N}}^{n+1}}{1_{\mathbb{C}} - z}$.

(37) If $z \neq 1_{\mathbb{C}}$ and for every n holds $s_1(n+1) = z \cdot s_1(n)$, then for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) = s_1(0) \cdot \frac{1_{\mathbb{C}} - z_{\mathbb{N}}^{n+1}}{1_{\mathbb{C}} - z}$.

3. Convergence of Complex Sequences

The following four propositions are true:

(38) Let a, b be sequences of real numbers and c be a complex sequence. Suppose that for every n holds $\Re(c(n)) = a(n)$ and $\Im(c(n)) = b(n)$. Then a is convergent and b is convergent if and only if c is convergent.

(39) Let a, b be convergent sequences of real numbers and c be a complex sequence. Suppose that for every n holds $\Re(c(n)) = a(n)$ and $\Im(c(n)) = b(n)$. Then c is convergent and $\lim c = \lim a + \lim b i$.

(40) Let a, b be sequences of real numbers and c be a convergent complex sequence. Suppose that for every n holds $\Re(c(n)) = a(n)$ and $\Im(c(n)) = b(n)$. Then a is convergent and b is convergent and $\lim a = \Re(\lim c)$ and $\lim b = \Im(\lim c)$.

(41) For every convergent complex sequence c holds $\Re(c)$ is convergent and $\Im(c)$ is convergent and $\lim \Re(c) = \Re(\lim c)$ and $\lim \Im(c) = \Im(\lim c)$.

Let c be a convergent complex sequence. Observe that $\Re(c)$ is convergent and $\Im(c)$ is convergent.

Next we state several propositions:

(42) Let c be a complex sequence. Suppose $\Re(c)$ is convergent and $\Im(c)$ is convergent. Then c is convergent and $\Re(\lim c) = \lim \Re(c)$ and $\Im(\lim c) = \lim \Im(c)$.

(43) If $0 < |z|$ and $|z| < 1$ and $s_1(0) = z$ and for every n holds $s_1(n+1) = s_1(n) \cdot z$, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

(44) If $|z| < 1$ and for every n holds $s_1(n) = z_{\mathbb{N}}^{n+1}$, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

(45) If $r > 0$ and there exists m such that for every n such that $n \geq m$ holds $|s_1(n)| \geq r$, then $|s_1|$ is not convergent or $\lim |s_1| \neq 0$.

(46) s_1 is convergent iff for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - s_1(n)| < p$.

(47) Suppose s_1 is convergent. Let given p. Suppose $0 < p$. Then there exists n such that for all natural numbers m, l such that $n \leq m$ and $n \leq l$ holds $|s_1(m) - s_1(l)| < p$.

(48) If for every n holds $|s_1(n)| \leq r_1(n)$ and r_1 is convergent and $\lim r_1 = 0$, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

4. Summable and Absolutely Summable Complex Sequences

Let s_1, s_2 be complex sequences. We say that s_1 is a subsequence of s_2 if and only if:

(Def. 9) There exists N_1 such that $s_1 = s_2\, N_1$.

Next we state three propositions:

(49) If s_1 is a subsequence of s_2, then $\Re(s_1)$ is a subsequence of $\Re(s_2)$ and $\Im(s_1)$ is a subsequence of $\Im(s_2)$.

(50) If s_1 is a subsequence of s_2 and s_2 is a subsequence of s_3, then s_1 is a subsequence of s_3.

(51) If s_1 is bounded, then there exists s_2 which is a subsequence of s_1 and convergent.

Let s_1 be a complex sequence. We say that s_1 is summable if and only if:

(Def. 10) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

Let us observe that there exists a complex sequence which is summable.
Let s_1 be a summable complex sequence. Observe that $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.
Let us consider s_1. We say that s_1 is absolutely summable if and only if:

(Def. 11) $|s_1|$ is summable.

Next we state the proposition

(52) If for every n holds $s_1(n) = 0_{\mathbb{C}}$, then s_1 is absolutely summable.

Let us mention that there exists a complex sequence which is absolutely summable.
Let s_1 be an absolutely summable complex sequence. Observe that $|s_1|$ is summable.
Next we state the proposition

(53) If s_1 is summable, then s_1 is convergent and $\lim s_1 = 0_{\mathbb{C}}$.

Let us note that every complex sequence which is summable is also convergent.
One can prove the following proposition

(54) If s_1 is summable, then $\Re(s_1)$ is summable and $\Im(s_1)$ is summable and $\sum s_1 = \sum \Re(s_1) + \sum \Im(s_1) i$.

Let s_1 be a summable complex sequence. Note that $\Re(s_1)$ is summable and $\Im(s_1)$ is summable.
The following propositions are true:

(55) If s_2 is summable and s_3 is summable, then $s_2 + s_3$ is summable and $\sum(s_2 + s_3) = \sum s_2 + \sum s_3$.

(56) If s_2 is summable and s_3 is summable, then $s_2 - s_3$ is summable and $\sum(s_2 - s_3) = \sum s_2 - \sum s_3$.

Let s_2, s_3 be summable complex sequences. One can check that $s_2 + s_3$ is summable and $s_2 - s_3$ is summable.
One can prove the following proposition

(57) If s_1 is summable, then $z\, s_1$ is summable and $\sum(z\, s_1) = z \cdot \sum s_1$.

Let z be an element of $\mathbb{C}$ and let s_1 be a summable complex sequence. One can check that $z\, s_1$ is summable.
Next we state two propositions:

(58) If $\Re(s_1)$ is summable and $\Im(s_1)$ is summable, then s_1 is summable and $\sum s_1 = \sum \Re(s_1) + \sum \Im(s_1) i$.

(59) If s_1 is summable, then for every n holds $s_1 \uparrow n$ is summable.

Let s_1 be a summable complex sequence and let n be a natural number. Note that $s_1 \uparrow n$ is summable.
We now state three propositions:

(60) If there exists n such that $s_1 \uparrow n$ is summable, then s_1 is summable.

(61) If s_1 is summable, then for every n holds $\sum s_1 = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + \sum(s_1 \uparrow (n+1))$.

(62) $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded iff s_1 is absolutely summable.

Let s_1 be an absolutely summable complex sequence. Observe that $(\sum_{\alpha=0}^{\kappa}|s_1|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded.
Next we state two propositions:

(63) s_1 is summable iff for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)| < p$.

(64) If s_1 is absolutely summable, then s_1 is summable.

Let us note that every complex sequence which is absolutely summable is also summable.
Let us note that there exists a complex sequence which is absolutely summable.
The following propositions are true:

(65) If $|z| < 1$, then $(z^{\kappa})_{\kappa\in\mathbb{N}}$ is summable and $\sum((z^{\kappa})_{\kappa\in\mathbb{N}}) = \frac{1_{\mathbb{C}}}{1_{\mathbb{C}}-z}$.

(66) If $|z| < 1$ and for every n holds $s_1(n+1) = z \cdot s_1(n)$, then s_1 is summable and $\sum s_1 = \frac{s_1(0)}{1_{\mathbb{C}}-z}$.

(67) If r_2 is summable and there exists m such that for every n such that $m \leq n$ holds $|s_3(n)| \leq r_2(n)$, then s_3 is absolutely summable.

(68) Suppose for every n holds $0 \leq |s_2|(n)$ and $|s_2|(n) \leq |s_3|(n)$ and s_3 is absolutely summable. Then s_2 is absolutely summable and $\sum |s_2| \leq \sum |s_3|$.

(69) If for every n holds $|s_1|(n) > 0$ and there exists m such that for every n such that $n \geq m$ holds $\frac{|s_1|(n+1)}{|s_1|(n)} \geq 1$, then s_1 is not absolutely summable.

(70) If for every n holds $r_2(n) = \sqrt[n]{|s_1|(n)}$ and r_2 is convergent and $\lim r_2 < 1$, then s_1 is absolutely summable.

(71) If for every n holds $r_2(n) = \sqrt[n]{|s_1|(n)}$ and there exists m such that for every n such that $m \leq n$ holds $r_2(n) \geq 1$, then $|s_1|$ is not summable.

(72) If for every n holds $r_2(n) = \sqrt[n]{|s_1|(n)}$ and r_2 is convergent and $\lim r_2 > 1$, then s_1 is not absolutely summable.

(73) Suppose $|s_1|$ is non-increasing and for every n holds $r_2(n) = 2^n \cdot |s_1|(2^n)$. Then s_1 is absolutely summable if and only if r_2 is summable.

(74) If $p > 1$ and for every n such that $n \geq 1$ holds $|s_1|(n) = \frac{1}{n^p}$, then s_1 is absolutely summable.

(75) If $p \leq 1$ and for every n such that $n \geq 1$ holds $|s_1|(n) = \frac{1}{n^p}$, then s_1 is not absolutely summable.

(76) If for every n holds $s_1(n) \neq 0_{\mathbb{C}}$ and $r_2(n) = \frac{|s_1|(n+1)}{|s_1|(n)}$ and r_2 is convergent and $\lim r_2 < 1$, then s_1 is absolutely summable.

(77) If for every n holds $s_1(n) \neq 0_{\mathbb{C}}$ and there exists m such that for every n such that $n \geq m$ holds $\frac{|s_1|(n+1)}{|s_1|(n)} \geq 1$, then s_1 is not absolutely summable.

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