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A Theory of Boolean Valued Functions and Quantifiers with Respect to Partitions

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Summary. In this paper, we define the coordinate of partitions. We also introduce the universal quantifier and the existential quantifier of Boolean valued functions with respect to partitions. Some predicate calculus formulae containing such quantifiers are proved. Such a theory gives a discussion of semantics to usual predicate logic.

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The articles [10], [3], [12], [16], [15], [13], [1], [7], [11], [14], [2], [8], [9], [6], [5], and [4] provide the notation and terminology for this paper.

1. Preliminaries

In this paper Y is a non empty set and G is a subset of PARTITIONS(Y).

Let X be a set. Then PARTITIONS(X) is a partition family of X.

Let X be a set and let F be a non empty partition family of X. We see that the element of F is a partition of X.

The following proposition is true

(1) Let y be an element of Y. Then there exists a subset X of Y such that

(i) $y \in X$, and

(ii) there exists a function h and there exists a family F of subsets of Y such that $\mathrm{dom}\, h = G$ and $\mathrm{rng}\, h = F$ and for every set d such that $d \in G$ holds $h(d) \in d$ and $X = \mathrm{Intersect}(F)$ and $X \neq \emptyset$.

Let us consider Y and let G be a subset of PARTITIONS(Y). The functor $\bigwedge G$ yields a partition of Y and is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in \bigwedge G$ if and only if there exists a function h and there exists a family F of subsets of Y such that $\mathrm{dom}\, h = G$ and $\mathrm{rng}\, h = F$ and for every set d such that $d \in G$ holds $h(d) \in d$ and $x = \mathrm{Intersect}(F)$ and $x \neq \emptyset$.

Let us consider Y, let G be a subset of PARTITIONS(Y), and let b be a set. We say that b is upper min depend of G if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every partition d of Y such that $d \in G$ holds b is a dependent set of d, and

(ii) for every set e such that $e \subseteq b$ and for every partition d of Y such that $d \in G$ holds e is a dependent set of d holds $e = b$.

The following proposition is true

(2) For every element y of Y such that $G \neq \emptyset$ there exists a subset X of Y such that $y \in X$ and X is upper min depend of G.

Let us consider Y and let G be a subset of PARTITIONS(Y). The functor $\bigvee G$ yields a partition of Y and is defined as follows:

(Def. 3)(i) For every set x holds $x \in \bigvee G$ iff x is upper min depend of G if $G \neq \emptyset$,

(ii) $\bigvee G = I(Y)$, otherwise.

We now state two propositions:

(3) For every subset G of PARTITIONS(Y) and for every partition P_1 of Y such that $P_1 \in G$ holds $P_1 \Supset \bigwedge G$.

(4) For every subset G of PARTITIONS(Y) and for every partition P_1 of Y such that $P_1 \in G$ holds $P_1 \Subset \bigvee G$.

2. COORDINATE AND QUANTIFIERS

Let us consider Y and let G be a subset of PARTITIONS(Y). We say that G is generating if and only if:

(Def. 4) $\bigwedge G = I(Y)$.

Let us consider Y and let G be a subset of PARTITIONS(Y). We say that G is independent if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let h be a function and F be a family of subsets of Y. Suppose $\operatorname{dom} h = G$ and $\operatorname{rng} h = F$ and for every set d such that $d \in G$ holds $h(d) \in d$. Then Intersect$(F) \neq \emptyset$.

Let us consider Y and let G be a subset of PARTITIONS(Y). We say that G is a coordinate if and only if:

(Def. 6) G is independent and generating.

Let us consider Y and let P_1 be a partition of Y. Then $\{P_1\}$ is a subset of PARTITIONS(Y).

Let us consider Y, let P_1 be a partition of Y, and let G be a subset of PARTITIONS(Y). The functor CompF(P_1, G) yielding a partition of Y is defined as follows:

(Def. 7) $\mathrm{CompF}(P_1, G) = \bigwedge G \setminus \{P_1\}$.

Let us consider Y, let a be an element of $Boolean^Y$, let G be a subset of PARTITIONS(Y), and let P_1 be a partition of Y. We say that a is independent of P_1, G if and only if:

(Def. 8) a is dependent of CompF(P_1, G).

Let us consider Y, let a be an element of $Boolean^Y$, let G be a subset of PARTITIONS(Y), and let P_1 be a partition of Y. The functor $\forall_{a,P_1} G$ yields an element of $Boolean^Y$ and is defined as follows:

(Def. 9) $\forall_{a,P_1} G = \mathrm{INF}(a, \mathrm{CompF}(P_1, G))$.

Let us consider Y, let a be an element of $Boolean^Y$, let G be a subset of PARTITIONS(Y), and let P_1 be a partition of Y. The functor $\exists_{a,P_1} G$ yields an element of $Boolean^Y$ and is defined as follows:

(Def. 10) $\exists_{a,P_1} G = \mathrm{SUP}(a, \mathrm{CompF}(P_1, G))$.

The following propositions are true:

(5) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{a,P_1}G$ is dependent of $\mathrm{CompF}(P_1,G)$.

(6) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\exists_{a,P_1}G$ is dependent of $\mathrm{CompF}(P_1,G)$.

(7) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{\mathit{true}(Y),P_1}G = \mathit{true}(Y)$.

(8) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\exists_{\mathit{true}(Y),P_1}G = \mathit{true}(Y)$.

(9) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{\mathit{false}(Y),P_1}G = \mathit{false}(Y)$.

(10) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\exists_{\mathit{false}(Y),P_1}G = \mathit{false}(Y)$.

(11) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{a,P_1}G \Subset a$.

(12) For every element a of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $a \Subset \exists_{a,P_1}G$.

(13) For all elements a, b of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{a\wedge b,P_1}G = \forall_{a,P_1}G \wedge \forall_{b,P_1}G$.

(14) For all elements a, b of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{a,P_1}G \vee \forall_{b,P_1}G \Subset \forall_{a\vee b,P_1}G$.

(15) For all elements a, b of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\forall_{a\Rightarrow b,P_1}G \Subset \forall_{a,P_1}G \Rightarrow \forall_{b,P_1}G$.

(16) For all elements a, b of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\exists_{a\vee b,P_1}G = \exists_{a,P_1}G \vee \exists_{b,P_1}G$.

(17) For all elements a, b of $\mathit{Boolean}^Y$ and for every subset G of $\mathrm{PARTITIONS}(Y)$ and for every partition P_1 of Y holds $\exists_{a\wedge b,P_1}G \Subset \exists_{a,P_1}G \wedge \exists_{b,P_1}G$.

(18) For all elements a, b of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\exists_{a,P_1}G \oplus \exists_{b,P_1}G \Subset \exists_{a\oplus b,P_1}G$.

(19) For all elements a, b of $\mathit{Boolean}^Y$ and for every partition P_1 of Y holds $\exists_{a,P_1}G \Rightarrow \exists_{b,P_1}G \Subset \exists_{a\Rightarrow b,P_1}G$.

In the sequel a, u denote elements of $\mathit{Boolean}^Y$.
Next we state a number of propositions:

(20) For every partition P_1 of Y holds $\neg\forall_{a,P_1}G = \exists_{\neg a,P_1}G$.

(21) For every partition P_1 of Y holds $\neg\exists_{a,P_1}G = \forall_{\neg a,P_1}G$.

(22) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{u\Rightarrow a,P_1}G = u \Rightarrow \forall_{a,P_1}G$.

(23) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{a\Rightarrow u,P_1}G = \exists_{a,P_1}G \Rightarrow u$.

(24) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{u\vee a,P_1}G = u \vee \forall_{a,P_1}G$.

(25) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{a\vee u,P_1}G = \forall_{a,P_1}G \vee u$.

(26) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{a\vee u,P_1}G \Subset \exists_{a,P_1}G \vee u$.

(27) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{u\wedge a,P_1}G = u \wedge \forall_{a,P_1}G$.

(28) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{a \wedge u, P_1} G = \forall_{a, P_1} G \wedge u$.

(29) For every partition P_1 of Y holds $\forall_{a \wedge u, P_1} G \Subset \exists_{a, P_1} G \wedge u$.

(30) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{u \oplus a, P_1} G \Subset u \oplus \forall_{a, P_1} G$.

(31) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{a \oplus u, P_1} G \Subset \forall_{a, P_1} G \oplus u$.

(32) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{u \Leftrightarrow a, P_1} G \Subset u \Leftrightarrow \forall_{a, P_1} G$.

(33) For every partition P_1 of Y such that u is independent of P_1, G holds $\forall_{a \Leftrightarrow u, P_1} G \Subset \forall_{a, P_1} G \Leftrightarrow u$.

(34) For every partition P_1 of Y such that u is independent of P_1, G holds $\exists_{u \vee a, P_1} G = u \vee \exists_{a, P_1} G$.

(35) For every partition P_1 of Y such that u is independent of P_1, G holds $\exists_{a \vee u, P_1} G = \exists_{a, P_1} G \vee u$.

(36) For every partition P_1 of Y such that u is independent of P_1, G holds $\exists_{u \wedge a, P_1} G = u \wedge \exists_{a, P_1} G$.

(37) For every partition P_1 of Y such that u is independent of P_1, G holds $\exists_{a \wedge u, P_1} G = \exists_{a, P_1} G \wedge u$.

(38) For every partition P_1 of Y holds $u \Rightarrow \exists_{a, P_1} G \Subset \exists_{u \Rightarrow a, P_1} G$.

(39) For every partition P_1 of Y holds $\exists_{a, P_1} G \Rightarrow u \Subset \exists_{a \Rightarrow u, P_1} G$.

(40) For every partition P_1 of Y such that u is independent of P_1, G holds $u \oplus \exists_{a, P_1} G \Subset \exists_{u \oplus a, P_1} G$.

(41) For every partition P_1 of Y such that u is independent of P_1, G holds $\exists_{a, P_1} G \oplus u \Subset \exists_{a \oplus u, P_1} G$.

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