JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

Oriented Metric-Affine Plane — Part I

Jarosław Zajkowski
Warsaw University
Białystok

Summary. We present (in Euclidean and Minkowskian geometry) definitions and some properties of oriented orthogonality relation. Next we consider consistence Euclidean space and consistence Minkowskian space.

MML Identifier: ANALORT.

WWW: http://mizar.org/JFM/Vol3/analort.html

The articles [6], [1], [2], [8], [7], [4], [3], and [5] provide the notation and terminology for this paper.

Let V be an Abelian non empty loop structure and let v, w be elements of V. Let us observe that the functor $v+w$ is commutative.

We adopt the following convention: V denotes a real linear space, u, u_1, u_2, v, v_1, v_2, w, w_1, x, y denote vectors of V, and n denotes a real number.

Let us consider V, x, y and let us consider u. The functor $\rho^{\rm M}_{x,y}(u)$ yields a vector of V and is defined by:

(Def. 1) $\rho^{\rm M}_{x,y}(u) = \pi^1_{x,y}(u)\cdot x + (-\pi^2_{x,y}(u))\cdot y$.

The following propositions are true:

(1) If x, y span the space, then $\rho^{\rm M}_{x,y}(u+v) = \rho^{\rm M}_{x,y}(u) + \rho^{\rm M}_{x,y}(v)$.

(2) If x, y span the space, then $\rho^{\rm M}_{x,y}(n\cdot u) = n\cdot\rho^{\rm M}_{x,y}(u)$.

(3) If x, y span the space, then $\rho^{\rm M}_{x,y}(0_V) = 0_V$.

(4) If x, y span the space, then $\rho^{\rm M}_{x,y}(-u) = -\rho^{\rm M}_{x,y}(u)$.

(5) If x, y span the space, then $\rho^{\rm M}_{x,y}(u-v) = \rho^{\rm M}_{x,y}(u) - \rho^{\rm M}_{x,y}(v)$.

(6) If x, y span the space and $\rho^{\rm M}_{x,y}(u) = \rho^{\rm M}_{x,y}(v)$, then $u = v$.

(7) If x, y span the space, then $\rho^{\rm M}_{x,y}(\rho^{\rm M}_{x,y}(u)) = u$.

(8) If x, y span the space, then there exists v such that $u = \rho^{\rm M}_{x,y}(v)$.

Let us consider V, x, y and let us consider u. The functor $\rho^{\rm E}_{x,y}(u)$ yields a vector of V and is defined by:

(Def. 2) $\rho^{\rm E}_{x,y}(u) = \pi^2_{x,y}(u)\cdot x + (-\pi^1_{x,y}(u))\cdot y$.

Next we state several propositions:

(9) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(-v) = -\rho^{\mathrm{E}}_{x,y}(v)$.

(10) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(u+v) = \rho^{\mathrm{E}}_{x,y}(u) + \rho^{\mathrm{E}}_{x,y}(v)$.

(11) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(u-v) = \rho^{\mathrm{E}}_{x,y}(u) - \rho^{\mathrm{E}}_{x,y}(v)$.

(12) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(n \cdot u) = n \cdot \rho^{\mathrm{E}}_{x,y}(u)$.

(13) If x, y span the space and $\rho^{\mathrm{E}}_{x,y}(u) = \rho^{\mathrm{E}}_{x,y}(v)$, then $u = v$.

(14) If x, y span the space, then $\rho^{\mathrm{E}}_{x,y}(\rho^{\mathrm{E}}_{x,y}(u)) = -u$.

(15) If x, y span the space, then there exists v such that $\rho^{\mathrm{E}}_{x,y}(v) = u$.

Let us consider V and let us consider x, y, u, v, u_1, v_1. We say that the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y if and only if:

(Def. 3) $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upharpoonleft\upharpoonright u_1, v_1$.

We say that the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y if and only if:

(Def. 4) $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upharpoonleft\upharpoonright u_1, v_1$.

We now state a number of propositions:

(16) If x, y span the space, then if $u, v \upharpoonleft\upharpoonright u_1, v_1$, then $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upharpoonleft\upharpoonright \rho^{\mathrm{E}}_{x,y}(u_1), \rho^{\mathrm{E}}_{x,y}(v_1)$.

(17) If x, y span the space, then if $u, v \upharpoonleft\upharpoonright u_1, v_1$, then $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upharpoonleft\upharpoonright \rho^{\mathrm{M}}_{x,y}(u_1), \rho^{\mathrm{M}}_{x,y}(v_1)$.

(18) Suppose x, y span the space. Suppose the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y. Then the segments v, v_1 and u_1, u are E-coherently orthogonal in the basis x, y.

(19) Suppose x, y span the space. Suppose the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y. Then the segments v, v_1 and u, u_1 are M-coherently orthogonal in the basis x, y.

(20) The segments u, u and v, w are E-coherently orthogonal in the basis x, y.

(21) The segments u, u and v, w are M-coherently orthogonal in the basis x, y.

(22) The segments u, v and w, w are E-coherently orthogonal in the basis x, y.

(23) The segments u, v and w, w are M-coherently orthogonal in the basis x, y.

(24) If x, y span the space, then u, v, $\rho^{\mathrm{E}}_{x,y}(u)$ and $\rho^{\mathrm{E}}_{x,y}(v)$ are orthogonal w.r.t. x, y.

(25) The segments u, v and $\rho^{\mathrm{E}}_{x,y}(u)$, $\rho^{\mathrm{E}}_{x,y}(v)$ are E-coherently orthogonal in the basis x, y.

(26) The segments u, v and $\rho^{\mathrm{M}}_{x,y}(u)$, $\rho^{\mathrm{M}}_{x,y}(v)$ are M-coherently orthogonal in the basis x, y.

(27) Suppose x, y span the space. Then $u, v \upharpoonleft\upharpoonright u_1, v_1$ if and only if there exist u_2, v_2 such that $u_2 \neq v_2$ and the segments u_2, v_2 and u, v are E-coherently orthogonal in the basis x, y and the segments u_2, v_2 and u_1, v_1 are E-coherently orthogonal in the basis x, y.

(28) Suppose x, y span the space. Then $u, v \upharpoonleft\upharpoonright u_1, v_1$ if and only if there exist u_2, v_2 such that $u_2 \neq v_2$ and the segments u_2, v_2 and u, v are M-coherently orthogonal in the basis x, y and the segments u_2, v_2 and u_1, v_1 are M-coherently orthogonal in the basis x, y.

(29) Suppose x, y span the space. Then u, v, u_1 and v_1 are orthogonal w.r.t. x, y if and only if one of the following conditions is satisfied:

(i) the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y, or

(ii) the segments u, v and v_1, u_1 are E-coherently orthogonal in the basis x, y.

(30) Suppose that

(i) x, y span the space,

(ii) the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y, and

(iii) the segments u, v and v_1, u_1 are E-coherently orthogonal in the basis x, y.

Then $u = v$ or $u_1 = v_1$.

(31) Suppose that

(i) x, y span the space,

(ii) the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y, and

(iii) the segments u, v and v_1, u_1 are M-coherently orthogonal in the basis x, y.

Then $u = v$ or $u_1 = v_1$.

(32) Suppose that

(i) x, y span the space,

(ii) the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y, and

(iii) the segments u, v and u_1, w are E-coherently orthogonal in the basis x, y.

Then

(iv) the segments u, v and v_1, w are E-coherently orthogonal in the basis x, y, or

(v) the segments u, v and w, v_1 are E-coherently orthogonal in the basis x, y.

(33) Suppose that

(i) x, y span the space,

(ii) the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y, and

(iii) the segments u, v and u_1, w are M-coherently orthogonal in the basis x, y.

Then

(iv) the segments u, v and v_1, w are M-coherently orthogonal in the basis x, y, or

(v) the segments u, v and w, v_1 are M-coherently orthogonal in the basis x, y.

(34) Suppose the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y. Then the segments v, u and v_1, u_1 are E-coherently orthogonal in the basis x, y.

(35) Suppose the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y. Then the segments v, u and v_1, u_1 are M-coherently orthogonal in the basis x, y.

(36) Suppose that

(i) x, y span the space,

(ii) the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y, and

(iii) the segments u, v and v_1, w are E-coherently orthogonal in the basis x, y.

Then the segments u, v and u_1, w are E-coherently orthogonal in the basis x, y.

(37) Suppose that

(i) x, y span the space,

(ii) the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y, and

(iii) the segments u, v and v_1, w are M-coherently orthogonal in the basis x, y.

Then the segments u, v and u_1, w are M-coherently orthogonal in the basis x, y.

(38) Suppose x, y span the space. Let given u, v, w. Then there exists u_1 such that $w \neq u_1$ and the segments w, u_1 and u, v are E-coherently orthogonal in the basis x, y.

(39) Suppose x, y span the space. Let given u, v, w. Then there exists u_1 such that $w \neq u_1$ and the segments w, u_1 and u, v are M-coherently orthogonal in the basis x, y.

(40) Suppose x, y span the space. Let given u, v, w. Then there exists u_1 such that $w \neq u_1$ and the segments u, v and w, u_1 are E-coherently orthogonal in the basis x, y.

(41) Suppose x, y span the space. Let given u, v, w. Then there exists u_1 such that $w \neq u_1$ and the segments u, v and w, u_1 are M-coherently orthogonal in the basis x, y.

(42) Suppose that

(i) x, y span the space,

(ii) the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y,

(iii) the segments w, w_1 and v, v_1 are E-coherently orthogonal in the basis x, y, and

(iv) the segments w, w_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y.

Then $w = w_1$ or $v = v_1$ or the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y.

(43) Suppose that

(i) x, y span the space,

(ii) the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y,

(iii) the segments w, w_1 and v, v_1 are M-coherently orthogonal in the basis x, y, and

(iv) the segments w, w_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y.

Then $w = w_1$ or $v = v_1$ or the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y.

(46)[1] Suppose that

(i) x, y span the space,

(ii) the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y,

(iii) the segments v, v_1 and w, w_1 are E-coherently orthogonal in the basis x, y, and

(iv) the segments u_2, v_2 and w, w_1 are E-coherently orthogonal in the basis x, y.

Then the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y or $v = v_1$ or $w = w_1$.

(47) Suppose that

(i) x, y span the space,

(ii) the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y,

(iii) the segments v, v_1 and w, w_1 are M-coherently orthogonal in the basis x, y, and

(iv) the segments u_2, v_2 and w, w_1 are M-coherently orthogonal in the basis x, y.

Then the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y or $v = v_1$ or $w = w_1$.

(48) Suppose that

(i) x, y span the space,

(ii) the segments u, u_1 and v, v_1 are E-coherently orthogonal in the basis x, y,

(iii) the segments v, v_1 and w, w_1 are E-coherently orthogonal in the basis x, y, and

(iv) the segments u, u_1 and u_2, v_2 are E-coherently orthogonal in the basis x, y.

Then the segments u_2, v_2 and w, w_1 are E-coherently orthogonal in the basis x, y or $v = v_1$ or $u = u_1$.

[1] The propositions (44) and (45) have been removed.

(49) Suppose that

(i) x, y span the space,

(ii) the segments u, u_1 and v, v_1 are M-coherently orthogonal in the basis x, y,

(iii) the segments v, v_1 and w, w_1 are M-coherently orthogonal in the basis x, y, and

(iv) the segments u, u_1 and u_2, v_2 are M-coherently orthogonal in the basis x, y.

Then the segments u_2, v_2 and w, w_1 are M-coherently orthogonal in the basis x, y or $v = v_1$ or $u = u_1$.

(50) Suppose x, y span the space. Let given v, w, u_1, v_1, w_1. Suppose that

(i) the segments v, v_1 and w, u_1 are not E-coherently orthogonal in the basis x, y,

(ii) the segments v, v_1 and u_1, w are not E-coherently orthogonal in the basis x, y, and

(iii) the segments u_1, w_1 and u_1, w are E-coherently orthogonal in the basis x, y.

Then there exists u_2 such that

(iv) the segments v, v_1 and v, u_2 are E-coherently orthogonal in the basis x, y or the segments v, v_1 and u_2, v are E-coherently orthogonal in the basis x, y, and

(v) the segments u_1, w_1 and u_1, u_2 are E-coherently orthogonal in the basis x, y or the segments u_1, w_1 and u_2, u_1 are E-coherently orthogonal in the basis x, y.

(51) Suppose x, y span the space. Then there exist u, v, w such that

(i) the segments u, v and u, w are E-coherently orthogonal in the basis x, y, and

(ii) for all v_1, w_1 such that the segments v_1, w_1 and u, v are E-coherently orthogonal in the basis x, y holds the segments v_1, w_1 and u, w are not E-coherently orthogonal in the basis x, y and the segments v_1, w_1 and w, u are not E-coherently orthogonal in the basis x, y or $v_1 = w_1$.

(52) Suppose x, y span the space. Let given v, w, u_1, v_1, w_1. Suppose that

(i) the segments v, v_1 and w, u_1 are not M-coherently orthogonal in the basis x, y,

(ii) the segments v, v_1 and u_1, w are not M-coherently orthogonal in the basis x, y, and

(iii) the segments u_1, w_1 and u_1, w are M-coherently orthogonal in the basis x, y.

Then there exists u_2 such that

(iv) the segments v, v_1 and v, u_2 are M-coherently orthogonal in the basis x, y or the segments v, v_1 and u_2, v are M-coherently orthogonal in the basis x, y, and

(v) the segments u_1, w_1 and u_1, u_2 are M-coherently orthogonal in the basis x, y or the segments u_1, w_1 and u_2, u_1 are M-coherently orthogonal in the basis x, y.

(53) Suppose x, y span the space. Then there exist u, v, w such that

(i) the segments u, v and u, w are M-coherently orthogonal in the basis x, y, and

(ii) for all v_1, w_1 such that the segments v_1, w_1 and u, v are M-coherently orthogonal in the basis x, y holds the segments v_1, w_1 and u, w are not M-coherently orthogonal in the basis x, y and the segments v_1, w_1 and w, u are not M-coherently orthogonal in the basis x, y or $v_1 = w_1$.

In the sequel u_3, v_3 are sets.

Let us consider V and let us consider x, y. The Euclidean oriented orthogonality defined over V, x, y yields a binary relation on [: the carrier of V, the carrier of V :] and is defined by the condition (Def. 5).

(Def. 5) The following statements are equivalent

(i) $\langle u_3, v_3 \rangle \in$ the Euclidean oriented orthogonality defined over V, x, y,

(ii) there exist u_1, u_2, v_1, v_2 such that $u_3 = \langle u_1, u_2 \rangle$ and $v_3 = \langle v_1, v_2 \rangle$ and the segments u_1, u_2 and v_1, v_2 are E-coherently orthogonal in the basis x, y.

Let us consider V and let us consider x, y. The Minkowskian oriented orthogonality defined over V, x, y yielding a binary relation on $[\!:$the carrier of V, the carrier of $V:\!]$ is defined by the condition (Def. 6).

(Def. 6) The following statements are equivalent

(i) $\langle u_3, v_3\rangle \in$ the Minkowskian oriented orthogonality defined over V, x, y,

(ii) there exist u_1, u_2, v_1, v_2 such that $u_3 = \langle u_1, u_2\rangle$ and $v_3 = \langle v_1, v_2\rangle$ and the segments u_1, u_2 and v_1, v_2 are M-coherently orthogonal in the basis x, y.

Let us consider V and let us consider x, y. The functor $\mathrm{CESpace}(V,x,y)$ yields a strict affine structure and is defined by:

(Def. 7) $\mathrm{CESpace}(V,x,y) = \langle$the carrier of V, the Euclidean oriented orthogonality defined over V, x, $y\rangle$.

Let us consider V and let us consider x, y. One can check that $\mathrm{CESpace}(V,x,y)$ is non empty.

Let us consider V and let us consider x, y. The functor $\mathrm{CMSpace}(V,x,y)$ yields a strict affine structure and is defined by:

(Def. 8) $\mathrm{CMSpace}(V,x,y) = \langle$the carrier of V, the Minkowskian oriented orthogonality defined over V, x, $y\rangle$.

Let us consider V and let us consider x, y. Observe that $\mathrm{CMSpace}(V,x,y)$ is non empty.

We now state two propositions:

(54) u_3 is an element of $\mathrm{CESpace}(V,x,y)$ iff u_3 is a vector of V.

(55) u_3 is an element of $\mathrm{CMSpace}(V,x,y)$ iff u_3 is a vector of V.

In the sequel p, q, r, s are elements of $\mathrm{CESpace}(V,x,y)$.

The following proposition is true

(56) Suppose $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$. Then $p,q \rceil\!|\!\lceil r,s$ if and only if the segments u, v and u_1, v_1 are E-coherently orthogonal in the basis x, y.

In the sequel p, q, r, s are elements of $\mathrm{CMSpace}(V,x,y)$.

Next we state the proposition

(57) Suppose $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$. Then $p,q \rceil\!|\!\lceil r,s$ if and only if the segments u, v and u_1, v_1 are M-coherently orthogonal in the basis x, y.

REFERENCES

[1] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical metric affine spaces and planes. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/analmetr.html.

[4] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/analoaf.html.

[5] Henryk Oryszczyszyn and Krzysztof Prażmowski. A construction of analytical ordered trapezium spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/geomtrap.html.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[7] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[8] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received October 24, 1991

Published January 2, 2004